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**NECESSARY AND SUFFICIENT CONDITIONS FOR PARETO  
OPTIMALITY IN INFINITE HORIZON COOPERATIVE  
DIFFERENTIAL GAMES**

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# Necessary and Sufficient Conditions for Pareto Optimality in Infinite Horizon Cooperative Differential Games

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## Abstract

In this article we derive necessary and sufficient conditions for the existence of Pareto optimal solutions for infinite horizon cooperative differential games. We consider games defined by non autonomous and discounted autonomous systems. The obtained results are used to analyze the regular indefinite linear quadratic infinite horizon differential game. For the scalar case, we present an algorithm, with mild conditions on the control space, to find all the Pareto optimal solutions.

**Keywords**    Pareto Efficiency, Cooperative Differential Games, Infinite Horizon Optimal Control, LQ theory.  
**JEL-Codes**    C61, C71, C73.

## 1 Introduction

In this paper we address the problem of finding the set of Pareto optimal solutions in the situation where a single player has multiple objectives or multiple players,  $N$  players here, decide to coordinate their actions with an intent to minimize their costs. The system or the dynamic environment where the players interact is modeled by a (set of) differential equation(s), and we assume an open-loop information structure. Every player  $i$  may choose his action/control trajectory,  $u_i(\cdot)$ , arbitrarily from the set  $\mathcal{U}^i$  of piecewise continuous functions<sup>1</sup>. Formally, the players are assumed to minimize the performance criteria:

$$J_i(t_0, x_0, u_1, u_2, \dots, u_N) = \int_{t_0}^{\infty} g_i(t, x(t), u_1(t), u_2(t), \dots, u_N(t)) dt, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the solution of the differential equation (dynamic environment)

$$\dot{x}(t) = f(t, x(t), u_1(t), u_2(t), \dots, u_N(t)), \quad x(t_0) = x_0 \in \mathbb{R}^n. \quad (2)$$

Here,  $u_i(t) \in \mathbb{R}^{m_i}$  with  $u_i(\cdot) \in \mathcal{U}^i$  and we denote  $u := (u_1, u_2, \dots, u_N) \in \mathcal{U}^1 \times \mathcal{U}^2 \times \dots \times \mathcal{U}^N = \mathcal{U}$  with  $\mathcal{U}$  being the set of admissible controls. Let  $m := m_1 + m_2 + \dots + m_N$ . For the above problem to be well-defined we assume that  $f(t, x, u) : \mathbb{R} \times \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}^n$  and  $g_i(t, x, u) : \mathbb{R} \times \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$ ,  $i = 1, 2, \dots, N$ , are continuous and all the partial derivatives of  $f$  and  $g_i$  w.r.t.  $x$  and  $u$  exist and are continuous. Further, we assume that the integrals involved in the player's objectives converge<sup>2</sup>.

Pareto optimality plays a central role in analyzing these problems. Since we are interested in the joint minimization of the objectives of the players, the cost incurred by a single player cannot be minimized without increasing the cost incurred by other players. So, we consider solutions which cannot be improved upon by all the players simultaneously; the so called Pareto optimal solutions. Formally, the set of controls  $u^* \in \mathcal{U}$  is Pareto optimal if the set of inequalities  $J_i(u) \leq J_i(u^*)$ ,  $i = 1, 2, \dots, N$ , with at least one of the inequalities being strict, does not allow for any solution in  $u \in \mathcal{U}$ . The corresponding point  $(J_1(u^*), J_2(u^*), \dots, J_N(u^*)) \in \mathbb{R}^N$  is called a Pareto solution. The set of all Pareto solutions is called the Pareto frontier.

<sup>1</sup>See, e.g., [15] or [7] for a generalization.

<sup>2</sup>If the integrals do not converge there exist other notions of optimality, see [23], [5], and the analysis becomes involved.

In this article we are interested in finding Pareto optimal solutions of the infinite horizon cooperative game problem (1,2). Here, we do not consider formation of sub coalitions and the possibility of utility transfers during the course of the game. We assume players make binding agreements towards cooperation at the start of the game and continue for ever. So, by varying the controls/actions in  $\mathcal{U}$  one obtains a set of feasible points in  $\mathbb{R}^N$  and the Pareto frontier constitutes the set of non improvable points. Further, we do not consider the aspect of selecting a particular point on the Pareto frontier, i.e., bargaining, and one may consult, e.g., chapter 6 of [7] for these issues. So, in cooperative game theory terminology the problem (1,2) relates to the issue of finding costs incurred by the grand coalition in a non transferable utility game described in strategic form, see [20] and [18].

A well known way to find Pareto optimal controls is to solve a parametrized optimal control problem [29, 16]. However, it is unclear whether all Pareto optimal solutions are obtained using this procedure, see example 4.2 [9]. The closest references we could track, towards finding Pareto solutions in differential games, are [6], [4] and [26]. The necessary conditions for Pareto solutions where cost functions are just functions of the terminal state were given in [26] and the affiliated papers [25] and [17]. In [26] geometric properties of Pareto surfaces were used to derive necessary conditions which are in the spirit of maximum principle. Some difference with our work are: they assume that the admissible controls are of the feedback type and the terminal state should belong to some  $n - 1$  dimensional surface. Recently, [9] gives necessary and sufficient conditions for Pareto optimality for finite horizon cooperative differential games.

Almost all of the earlier works address the problem of finding Pareto solutions in the finite horizon case. In this article we focus on infinite horizon cooperative differential games. In section 2 we present a necessary and sufficient characterization of Pareto optimality which entails to reformulate the Pareto optimality problem as  $N$  constrained infinite horizon optimal control problems. As a consequence, our results are in the spirit of the maximum principle. We stress that this reformulation should not be confused with decentralization problems where each player  $i$  by choosing actions, without coordination, from the strategy set  $\mathcal{U}^i$  to achieve a Pareto optimum. Instead, in our approach, the constraints depend upon a Pareto solution due to the above reformulation, see lemma 2.2.

Due to the above reformulation our results are closely related to necessary and sufficient conditions for optimality of infinite horizon optimal control problems. Infinite horizon optimal control problems arise when no natural bound can be placed on the time horizon, for example while modeling capital accumulation processes (economic growth) and in biological sciences. In his seminal work on the theory of saving, Ramsey [21] used a dynamic optimization model defined on an unbounded time horizon, see [1] for more details. As the objectives can grow unbounded different notions of optimality have been introduced, see [5, 23] more details on the analysis. The necessary conditions for optimality given by the maximum principle are incomplete as transversality conditions are not clearly specified. As a result one obtains a large number of extremal trajectories. A natural extension of finite horizon transversality conditions, in general, is not possible, see [12]. Only by imposing certain restrictions on the system such an extension can be made, see [19], [23], [22] and more recently [2] and [27].

In section 3 we show, by making a particular assumption on the Lagrange multipliers, that the necessary conditions for Pareto optimality are same as the necessary conditions for optimality of a weighted sum optimal control problem. Further, we observe that an extension of finite horizon transversality conditions is a weak sufficient condition to satisfy this assumption. For discounted autonomous systems, [19] derives necessary conditions for free endpoint optimal control problems. We extend these results for the constrained problems (due to the above reformulation) and derive weak sufficient conditions for this assumption to hold true. In section 4 we derive sufficient conditions for Pareto optimality in the spirit of Arrow's sufficiency results in optimal control. In section 5 we consider regular indefinite infinite planning horizon linear quadratic differential games where the cost involved for the state variable has an arbitrary sign and the use of every control is quadratically penalized. We observe that if the dynamic system is controllable then this assumption holds true naturally. The linear quadratic case was recently solved for both a finite and infinite planning horizon in [8] assuming that the problem is a convex function of the control variables and the initial state is arbitrary. In this article we concentrate on the general case and where the initial state is fixed and the planning horizon is infinite. We provide some examples to illustrate subtleties and open issues. For the scalar case, by imposing a restriction on the control space, we show that all Pareto optimal solutions can be obtained using the weighting method and provide an algorithm to compute all the Pareto solutions.

**Notation:** We use the following notation. Let  $\bar{N} = \{1, 2, \dots, N\}$  denote the grand coalition and let  $\bar{N} \setminus \{i\}$  denote the coalition of all players excluding player  $i$ . Let  $\mathcal{P}_N$  denote the  $N$  dimensional unit simplex.  $\mathbb{R}_+^N$  denotes a cone consisting of  $N$  dimensional vectors with non negative entries.  $\mathbf{1}_N$  denotes a vector in  $\mathbb{R}^N$  with all its entries equal to 1.  $y'$  represents the transpose of the vector  $y \in \mathbb{R}^N$ .  $|x|$  represents the absolute value of  $x \in \mathbb{R}$ .  $\|y\|$  represents the Euclidean norm of the vector  $y \in \mathbb{R}^N$ .  $|y|_i$  represents the absolute value of the  $i^{th}$  entry of the vector  $y$ .  $|A|_{(m,n)}$  represents the absolute value of entry  $(m, n)$  of the matrix  $A$ .  $A > 0$  denotes matrix  $A$  is strictly positive definite.  $f_x(\cdot)$

represents the partial derivative of the function  $f(\cdot)$  w.r.t  $x$ .  $\Phi_{f_x}(t, 0) = e^{\int_0^t f_x(x, u) dt}$  represents the state transition matrix associated with the linear autonomous linear ordinary differential equation  $\dot{x} = f_x(x, u)x$ ,  $x(0) = x_0$ .  $\vec{\omega} \in \mathbb{R}_+^N$  denotes the vector whose entries  $\omega_i$  are the weights assigned to the cost function of each player. We define the weighted sum function  $G(\cdot)$  as  $G(\vec{\omega}, t, x(t), u(t)) = \sum_{i \in \bar{N}} \omega_i g_i(t, x(t), u(t))$ . A matrix  $M_{\vec{\omega}}$  represents  $\sum_{i \in \bar{N}} \omega_i M_i$  where  $\vec{\omega}$  represents the weight vector.  $\text{sp}\{v_1, v_2, \dots, v_k\}$  represents the subspace spanned by the vectors  $v_1, v_2, \dots, v_k$ .

## 2 Pareto Optimality

In this section we state conditions to characterize Pareto optimal controls. Lemma 2.1, given below, states that every control minimizing a weighted sum of the cost function of all players (where all weights are strictly positive) is Pareto optimal. So, varying the positive weights over the unit simplex one obtains, in principle, different Pareto optimal controls. A proof of the lemma can be found in [9, 16].

**Lemma 2.1.** *Let  $\alpha_i \in (0, 1)$ , with  $\sum_{i=1}^N \alpha_i = 1$ . Assume  $u^* \in \mathcal{U}$  is such that*

$$u^* \in \arg \min_{u \in \mathcal{U}} \left\{ \sum_{i=1}^N \alpha_i J_i(u) \right\}. \quad (3)$$

*Then  $u^*$  is Pareto optimal.*

The above lemma implies that minimizing the weighted sum is an easy way to find Pareto optimal controls. Being a sufficient condition it is, however, unclear whether we obtain all Pareto optimal controls in this way. Lemma 2.2 mentioned below gives both a necessary and sufficient characterization of Pareto solutions. It states that every player's Pareto optimal solutions can be obtained as the solution of a constrained optimization problem. The proof is along the lines of the finite dimensional case considered in chapter 22 of [24].

**Lemma 2.2.**  *$u^* \in \mathcal{U}$  is Pareto optimal if and only if for all  $i$ ,  $u^*(\cdot)$  minimizes  $J_i(u)$  on the constrained set*

$$\mathcal{U}_i = \{u \mid J_j(u) \leq J_j(u^*), j = 1, \dots, N, j \neq i\}, \text{ for } i = 1, \dots, N. \quad (4)$$

*Proof.*  $\Rightarrow$  Suppose  $u^*$  is Pareto optimal. Then  $u^* \in \mathcal{U}_k, \forall k$ , so  $\mathcal{U}_k \neq \emptyset$ . Now, if  $u^*$  does not minimize  $J_k(u)$  on the constraint set  $\mathcal{U}_k$  for some  $k$ , then there exists a  $u$  such that  $J_j(u) \leq J_j(u^*)$  for all  $j \neq k$  and  $J_k(u) < J_k(u^*)$ . This contradicts the Pareto optimality of  $u^*$ .

$\Leftarrow$  Suppose  $u^*$  minimizes each  $J_k(u)$  on  $\mathcal{U}_k$ . If  $\hat{u}$  does not provide a Pareto optimum, then there exists a  $u(\cdot) \in \mathcal{U}$  and an index  $k$  such that  $J_i(u) \neq J_i(u^*)$  for all  $i$  and  $J_k(u) < J_k(u^*)$ . This contradicts the minimality of  $u^*$  for  $J_k(u)$  on  $\mathcal{U}_k$ .  $\square$

We observe that for a fixed player the constraint set  $\mathcal{U}_i$  defined in (4) depends on the entries of the Pareto optimal solution that represents the loss of the other players. Therefore this result mainly serves theoretical purposes, as we will see, e.g., in the proof of theorem 3.1 and theorem 3.3. Using the above lemma, we next argue that Pareto optimal controls satisfy the dynamic programming principle.

**Corollary 2.1.** *If  $u^* \in \mathcal{U}$  is a Pareto optimal control for  $x(0) = x_0$  in (1,2), then for any  $\tau > 0$ ,  $u^*([ \tau, \infty ))$  is a Pareto optimal control for  $x(\tau) = x^*(\tau)$  in (1,2). Here,  $x^*(\tau) = x(t, 0, u^*([0, \tau]))$  is the value of the state at  $\tau$  generated by  $u^*([0, \tau])$ .*

*Proof.* Let  $\mathcal{U}_i(\tau)$ , with  $x(\tau) = x^*(\tau)$ , be the constrained set defined as:

$$\mathcal{U}_i(\tau) := \{u \mid J_j(x(\tau), u) \leq J_j(x(\tau), u^*([ \tau, \infty ))), j = 1, \dots, N, j \neq i\}.$$

Consider a control  $u \in \mathcal{U}_i(\tau)$  and let  $u^e([0, \infty))$  be a control defined on  $[0, \infty)$  such that  $u^e([0, \tau)) = u^*([0, \tau))$  and  $u^e([ \tau, \infty )) = u$ , then  $x(\tau, 0, u^e([0, \tau)) = x^*(\tau)$ . Further,

$$\begin{aligned} J_j(0, x_0, u^e) &= \int_0^\infty g_j(t, x(t), u^e(t)) dt \\ &= \int_0^\tau g_j(t, x^*(t), u^*(t)) dt + \int_\tau^\infty g_j(t, x(t), u(t)) dt \\ &\quad (\text{as } u \in \mathcal{U}_i(\tau), \text{ we have}) \\ &\leq \int_0^\tau g_j(t, x^*(t), u^*(t)) dt + \int_\tau^\infty g_j(t, x^*(t), u^*(t)) dt \\ &= \int_0^\infty g_j(t, x^*(t), u^*(t)) dt = J_j(0, x_0, u^*). \end{aligned}$$

The above inequality holds for all  $j = 1, \dots, N$ ,  $j \neq i$ . Clearly,  $u^e([0, \infty)) \in \mathcal{U}_i(0)$  i.e., every element  $u \in \mathcal{U}_i(\tau)$  can be viewed as an element  $u^e \in \mathcal{U}_i(0)$  restricted to the time interval  $[\tau, \infty)$ . From the dynamic programming principle it follows directly that  $u^*([\tau, \infty))$  has to minimize  $J_i(\tau, x^*(\tau), u)$  on  $\mathcal{U}_i(\tau)$ .  $\square$

Another result that follows directly from lemma 2.2 is that if the argument at which some player's cost is minimized is unique, then this control is Pareto optimal too (see corollary 2.5 in [9] for the proof).

**Corollary 2.2.** *Assume  $J_1(u)$  has a minimum which is uniquely attained at  $u^*$ . Then  $(J_1(u^*), J_2(u^*), \dots, J_N(u^*))$  is a Pareto solution.*

We give the following result from lemma 2.2. Since, Pareto optimality is preserved for every strictly monotonic transformation of the cost functions, if the player's costs are modified as  $\tilde{J}_i(u) = J_i(u) - c$ ,  $c \in \mathbb{R}$ ,  $\forall i \in \bar{N}$ , then we have the following corollary.

**Corollary 2.3.** *The set of Pareto optimal strategies for the games with player's objectives as  $J_i(u)$  and  $\tilde{J}_i(u)$ ,  $i \in \bar{N}$ , is the same.*

### 3 Necessary Conditions for the General Case

In this section, using lemma 2.2 we derive necessary conditions of Pareto optimality for the problem (2,1) in a general setting. Before proceeding in this direction we give the following notation for the  $N$  person infinite horizon cooperative differential game:

$$(P) \quad \begin{aligned} & \text{for each } i \in \bar{N} \quad \min_{u \in \mathcal{U}} \int_0^\infty g_i(t, x(t), u(t)) dt \\ & \text{sub. to } \dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0. \end{aligned}$$

Let  $u^*$  be a Pareto optimal strategy for the problem (P) and  $x^*$  be the trajectory generated by  $u^*$ . Using lemma 2.2, (P) is equivalently written as  $N$  constrained optimal control problems, denoted by  $(P_i)$  for each player  $i \in \bar{N}$ , as follows:

$$(P_i) \quad \begin{aligned} & \min_{u \in \mathcal{U}_i} \int_0^\infty g_i(t, x(t), u(t)) dt \\ & \text{sub. to } \dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0. \end{aligned}$$

The control space  $\mathcal{U}_i$  in  $(P_i)$  is constrained and depends on the Pareto optimal solution of players  $j \in \bar{N} \setminus \{i\}$ . Introducing the auxiliary states  $\tilde{x}_j^i(t)$ ,  $j \in \bar{N} \setminus \{i\}$  as

$$\tilde{x}_j^i(t) = \int_0^t g_j(t, x(t), u(t)) dt, \quad \tilde{x}_j^i(0) = 0,$$

the constraint set  $\mathcal{U}_i$  can be represented as

$$\mathcal{U}_i = \left\{ u \in \mathcal{U} \mid \dot{\tilde{x}}_j^i(t) = g_j(t, x(t), u(t)), \tilde{x}_j^i(0) = 0, \lim_{t \rightarrow \infty} \tilde{x}_j^i(t) \leq \tilde{x}_j^{i*} = \int_0^\infty g_j(t, x^*(t), u^*(t)) dt, \forall j \in \bar{N} \setminus \{i\} \right\}.$$

The unconstrained representation, w.r.t to control space, of  $(P_i)$  is then given as:

$$(P_i) \quad \begin{aligned} & \min_{u \in \mathcal{U}} \int_0^\infty g_i(t, x(t), u(t)) dt \\ & \text{sub. to } \dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0 \\ & \quad \dot{\tilde{x}}_j^i(t) = g_j(t, x(t), u(t)), \quad \tilde{x}_j^i(0) = 0, \quad \lim_{t \rightarrow \infty} \tilde{x}_j^i(t) \leq \tilde{x}_j^{i*}, \quad \forall j \in \bar{N} \setminus \{i\}. \end{aligned}$$

Collecting the above and from lemma 2.2, we have the following proposition.

**Proposition 3.1.**  *$u^*$  is a Pareto optimal control for the cooperative game problem (P)  $\Leftrightarrow u^*$  is an optimal control for the problems  $(P_i)$ ,  $i \in \bar{N}$ .*

The optimal control problems  $(P_i)$  have mixed end point constraints, i.e.,  $\lim_{t \rightarrow \infty} x(t)$  is free and  $\lim_{t \rightarrow \infty} \tilde{x}_j^i(t)$ ,  $j \in \bar{N} \setminus \{i\}$  are constrained. Let  $H_i$  denote the Hamiltonian associated with the problem  $(P_i)$  and be defined as (with abuse of notation)  $H_i := \lambda_i^0 g_i + \lambda_i' f + \sum_{j \in \bar{N} \setminus \{i\}} \mu_j^i g_j$ . From proposition 3.1, by applying Pontryagin maximum principle for

( $P_i$ ) one can obtain necessary conditions for Pareto optimality for ( $P$ ). These necessary conditions give a set of extremal trajectories and the associated transversality conditions allow one to single out the optimal one. If the game problem ( $P$ ) is finite horizon type, then the transversality conditions associated with the problem ( $P_i$ ) are  $\lambda_i(T) = 0$ ,  $0 < T < \infty$  and  $\mu_j^i \geq 0$  for  $j \in \bar{N} \setminus \{i\}$ ,  $i \in \bar{N}$  and the maximum principle holds in normal form i.e.,  $\lambda_i^0 = 1$ ,  $i \in \bar{N}$  (refer to proposition 3.16, [11]). Using these ideas it was shown in [9] that necessary conditions for Pareto optimality for ( $P$ ) are same as the necessary conditions for optimality of a weighted sum optimal control problem. The main result there hinges upon the transversality conditions and normality of the problems ( $P_i$ ). Unfortunately, for the infinite horizon case the necessary conditions for optimality of the problems ( $P_i$ ) are incomplete (see pg. 234, theorem 12 of [23]). The above finite horizon transversality conditions generally do not naturally carry over to the infinite horizon case. Refer to [12, 11, 19, 23, 2] for counterexamples to illustrate this behavior.

We will see in the following discussion that  $\mu_j^i(t) = \mu_j^i$  (constants) due to the special structure of  $x_j^i(t)$ . Let  $\vec{\lambda}_i = (\mu_1^i, \dots, \mu_{i-1}^i, \lambda_i^0, \mu_{i+1}^i, \dots, \mu_N^i)'$ ,  $i \in \bar{N}$ . In the theorem 3.1 below, by making an assumption on  $\vec{\lambda}_i$  we show, using proposition 3.1, that necessary conditions of Pareto optimality of ( $P$ ) are the same as the necessary conditions for optimality of a weighted sum optimal control problem.

**Assumption 1.** For each problem ( $P_i$ ), the Lagrange multipliers associated with the objective function and the states ( $\vec{x}_j^i(t)$ ) are non negative with at least one of them strictly positive, i.e.,  $\vec{\lambda}_i \in \mathbb{R}_+^N \setminus \{0\}$ .

**Theorem 3.1.** If  $(J_1(u^*), J_2(u^*), \dots, J_N(u^*))$  is a Pareto candidate for problem ( $P$ ) and assumption (1) holds, then there exists an  $\vec{\alpha} \in \mathcal{P}_N$ , a co-state function  $\lambda(t) : [0, \infty) \rightarrow \mathbb{R}^n$  such that with  $H(\vec{\alpha}, t, x(t), u(t), \lambda(t)) = \lambda'(t)f(t, x(t), u(t)) + G(\vec{\alpha}, t, x(t), u(t))$ , the following conditions are satisfied.

$$H(\vec{\alpha}, t, x^*(t), u^*(t), \lambda(t)) \leq H(\vec{\alpha}, t, x^*(t), u(t), \lambda(t)) \quad (5a)$$

$$H^0(\vec{\alpha}, t, x^*(t), \lambda(t)) = \min_{u(t)} H(\vec{\alpha}, t, x^*(t), u(t), \lambda(t))$$

$$\dot{\lambda}(t) = -H_x^0(\vec{\alpha}, t, x^*(t), \lambda(t)) \quad (5b)$$

$$x^*(t) = H_\lambda^0(\vec{\alpha}, t, x^*(t), \lambda(t)) \quad \text{s.t. } x^*(0) = x_0 \quad (5c)$$

$$(\vec{\alpha}, \lambda(t)) \neq 0, \forall t \in [0, \infty), \vec{\alpha} \in \mathcal{P}_N. \quad (5d)$$

*Proof.* From proposition 3.1, if  $u^*$  is Pareto optimal for ( $P$ ) then the pair  $(x^*, u^*)$  is optimal for the problem ( $P_i$ ). We define the Hamiltonian associated with ( $P_i$ ) as:

$$H_i(t, x(t), u(t), \lambda(t)) = \lambda_i'(t)f(t, x(t), u(t)) + \lambda_i^0 g_i(t, x(t), u(t)) + \sum_{j \in \bar{N} \setminus \{i\}} \mu_j^i(t) g_j(t, x(t), u(t)). \quad (6)$$

So, from Pontryagin's maximum principle there exist a constant  $\lambda_i^0$  and co-state functions (continuous and piecewise continuously differentiable)  $\lambda_i(t) \in \mathbb{R}^n$  and  $\mu_j^i(t) \in \mathbb{R}$ ,  $j \in S_N^i$  such that:

$$(\lambda_i^0, \lambda_i(t), \mu_j^i(t)) \neq (0, 0, 0), j \in \bar{N} \setminus \{i\}, t \in [0, \infty) \quad (7a)$$

$$H_i(t, x^*(t), u^*(t), \lambda(t)) \leq H_i(t, x^*(t), u(t), \lambda(t)) \quad (7b)$$

$$H_i^0(t, x(t), \lambda(t)) = \min_{u(t)} H_i(t, x(t), u(t), \lambda(t))$$

$$\dot{\lambda}_i(t) = -H_{i,x}^0(t, x^*(t), \lambda(t)) \quad (7c)$$

$$\dot{\mu}_j^i(t) = -H_{i,x_j}^0(t, x^*(t), \lambda(t)). \quad (7d)$$

Since  $(H_i^0)_{x_j^i} = 0$ , multipliers associated with the auxiliary variables  $\mu_j^i(t) = \mu_j^i$  (constants) and the Hamiltonian can be written as  $H_i(t, x(t), u(t), \lambda(t)) = \lambda_i'(t)f(t, x(t), u(t)) + G(\vec{\lambda}_i, t, x(t), u(t))$ . The first order conditions are:

$$\lambda_i'(t)f(t, x^*(t), u^*(t)) + G(\vec{\lambda}_i, t, x^*(t), u^*(t)) \leq \lambda_i'(t)f(t, x^*(t), u(t)) + G(\vec{\lambda}_i, t, x^*(t), u(t)) \quad (8)$$

$$\dot{\lambda}_i(t) = -f_x'(t, x^*(t), u^*(t))\lambda_i(t) - G_x(\vec{\lambda}_i, t, x^*(t), u^*(t)). \quad (9)$$

Taking a sum over  $i \in \bar{N}$  for (8) and (9) yields

$$\sum_{i \in \bar{N}} \left( \lambda_i'(t)f(t, x^*(t), u^*(t)) + G(\vec{\lambda}_i, t, x^*(t), u^*(t)) \right) \leq \sum_{i \in \bar{N}} \left( \lambda_i'(t)f(t, x^*(t), u(t)) + G(\vec{\lambda}_i, t, x^*(t), u(t)) \right), \quad (10)$$

$$\sum_{i \in \bar{N}} \dot{\lambda}_i(t) = -f'_x(t, x^*(t), u^*(t)) \sum_{i \in \bar{N}} \lambda_i(t) - \sum_{i \in \bar{N}} G_x(\vec{\lambda}_i, t, x^*(t), u^*(t)). \quad (11)$$

Let us introduce  $d := \sum_{i \in \bar{N}} (\lambda_i^0 + \sum_{j \in \bar{N} \setminus \{i\}} \mu_j^i)$ . By assumption 1 we have  $d > 0$ . We define  $\lambda(t) := \frac{1}{d} \sum_{i \in \bar{N}} \lambda_i(t)$ ,  $\alpha_i := \frac{1}{d} (\lambda_i^0 + \sum_{j \in \bar{N} \setminus \{i\}} \mu_j^i)$ ,  $i \in \bar{N}$  and a vector  $\vec{\alpha} := (\alpha_1, \dots, \alpha_N)'$ . Notice that  $\vec{\alpha} \in \mathcal{P}_N$  by assumption (1). Dividing the equation (10) by  $d$  we have:

$$\lambda'(t)f(t, x^*(t), u^*(t)) + G(\vec{\alpha}, t, x^*(t), u^*(t)) \leq \lambda'(t)f(t, x^*(t), u(t)) + G(\vec{\alpha}, t, x^*(t), u(t)) \quad (12)$$

$$\dot{\lambda}(t) = -f'_x(t, x^*(t), u^*(t))\lambda(t) - G_x(\vec{\alpha}, t, x^*(t), u^*(t)). \quad (13)$$

Next we define the modified Hamiltonian as

$$H(\vec{\alpha}, t, x(t), u(t), \lambda(t)) = \lambda'(t)f(t, x(t), u(t)) + G(\vec{\alpha}, t, x(t), u(t)).$$

The above necessary conditions for  $u^*$  to be Pareto optimal control can be rewritten as (5).  $\square$

*Remark 1.* The necessary conditions given by (5) are closely related to the minimization of  $\sum_{i \in \bar{N}} \alpha_i J_i$  subject to (2), i.e., the weighted sum optimal control problem. There are, however, some subtle differences. When the weighted sum optimal control problem admits maximum principle in normal form then one obtains necessary conditions as (5).

A natural extension of finite horizon transversality conditions to the infinite horizon case for the problem  $(P_i)$ ,  $i \in \bar{N}$  leads to  $\lambda_i^0 = 1$  and  $\mu_j^i \geq 0$  for  $i \in \bar{N}$ , and as result guarantees assumption 1. For the analysis that follows from now onwards we focus on weak sufficient conditions that allow such an extension. Towards that end, we first consider non-autonomous systems. In general, such an extension is achieved by imposing restrictions on the system parameters, also called as growth conditions. Specializing theorem 3.16 [23] or example 10.3 [22] to the problem  $(P_i)$ , we have the following corollary:

**Corollary 3.1.** Suppose  $\int_0^\infty g_i(t, x(t), u(t))dt > -\infty$ ,  $i \in \bar{N}$  and there exist non-negative numbers  $a$ ,  $b$  and  $c$  with  $c > Nb$  such that the following conditions are satisfied for  $t \geq 0$  and all  $x(t)$ :

$$(|g_{i_x}(t, x(t), u(t))|)_m \leq ae^{-ct}, \quad m = 1, \dots, N, \quad \forall i \in \bar{N} \quad (14a)$$

$$(|f_x(t, x(t), u(t))|)_{(l,m)} \leq b, \quad l = 1, \dots, N, \quad m = 1, \dots, N. \quad (14b)$$

Then assumption 1 is satisfied. Consequently, for every Pareto solution the necessary conditions given by (5) hold true and in addition  $\lim_{t \rightarrow \infty} \lambda(t) = 0$  is satisfied.

*Proof.* If conditions (14) hold true, then by theorem 3.16 of [23], the finite horizon transversality conditions do extend to the infinite horizon case. As a result,  $\lambda_i^0 = 1$ ,  $\mu_j^i \geq 0$ ,  $\forall j \in \bar{N} \setminus \{i\}$  and  $\lim_{t \rightarrow \infty} \lambda_i(t) = 0$  are satisfied for the constrained optimal control problem  $(P_i)$  we have  $\vec{\lambda}_i \in \mathbb{R}_+^N \setminus \{0\}$ . Clearly assumption (1) is satisfied. So, the necessary conditions given by (5) hold true and in addition  $\lim_{t \rightarrow \infty} \lambda(t) = 0$ .  $\square$

The following example demonstrates the application of theorem 3.1 and corollary 3.1.

*Example 1.* Consider the following game problem:

$$\begin{aligned} (P) \quad & J_1(x_0, u_1, u_2) = \int_0^\infty e^{-\rho t/2} (u_1(t) - u_2(t)) dt \quad \text{and} \\ & J_2(x_0, u_1, u_2) = \int_0^\infty e^{-3\rho t/2} x^2(t) (u_2(t) - u_1(t)) dt \\ \text{sub. to} \quad & \dot{x}(t) = \frac{\rho}{2} x(t) + u_1(t) - u_2(t), \quad x(0) = 0, \quad t \in [0, \infty) \\ & u \in \mathcal{U}, \text{ s.t. } \mathcal{U} = \{u(\cdot) \mid \forall t \geq 0, u(t) \in E \subset \mathbb{R}^m, E \text{ is a closed and bounded set}\} \end{aligned} \quad (15)$$

Taking the transformations  $\tilde{x}(t) = e^{-\rho t/2} x(t)$  and  $\tilde{u}_i(t) = e^{-\rho t/2} u_i(t)$ , we transform the game  $(P)$  as a new game  $(\tilde{P})$  given by:

$$\begin{aligned} (\tilde{P}) \quad & J_1(x_0, \tilde{u}_1, \tilde{u}_2) = \int_0^\infty (\tilde{u}_1(t) - \tilde{u}_2(t)) dt \quad \text{and} \\ & J_2(x_0, \tilde{u}_1, \tilde{u}_2) = \int_0^\infty \tilde{x}^2(t) (\tilde{u}_2(t) - \tilde{u}_1(t)) dt \\ \text{sub. to} \quad & \dot{\tilde{x}}(t) = \tilde{u}_1(t) - \tilde{u}_2(t), \quad \tilde{x}(0) = 0, \quad t \in [0, \infty) \\ & \tilde{u}_i(t) = e^{-\rho t/2} u_i(t), \quad i = 1, 2. \end{aligned} \quad (16)$$



We have  $J_i(x_0, u_1, u_2) = J_i(x_0, \tilde{u}_1, \tilde{u}_2)$ ,  $i = 1, 2$ . The player's objectives are simplified as:

$$\begin{aligned} J_1(x_0, \tilde{u}_1, \tilde{u}_2) &= \int_0^\infty (\tilde{u}_1(t) - \tilde{u}_2(t)) dt = \lim_{t \rightarrow \infty} \tilde{x}(t) \\ J_2(x_0, \tilde{u}_1, \tilde{u}_2) &= \int_0^\infty \tilde{x}^2(t) (\tilde{u}_2(t) - \tilde{u}_1(t)) dt = -\frac{1}{3} \left( \lim_{t \rightarrow \infty} \tilde{x}(t) \right)^3. \end{aligned}$$

By construction,  $|u(t)| < c$  for some  $c > 0$ ,  $\forall t \geq 0$  and  $\lim_{t \rightarrow \infty} \tilde{x}(t) \leq 2c/\rho$ . We notice that  $J_2 = -\frac{1}{3}J_1^3$  for all  $(u_1, u_2)$  and choosing different values for the control functions  $u_i(\cdot)$ , every point in the  $(J_1, J_2)$  plane satisfying  $J_2 = -\frac{1}{3}J_1^3$  can be attained. Moreover, every point on this curve is Pareto optimal. This conclusion can be derived from the application of theorem 3.1 and corollary 3.1 too. With straightforward calculations we can show that for  $(\tilde{P})$ ,  $|f_{\tilde{x}}(\cdot)| = 0$ ,  $|g_{1\tilde{x}}(\cdot)| = 0$ ,  $|g_{2\tilde{x}}(\cdot)| \leq 2|\tilde{x}(t)||\tilde{u}_1(t) - \tilde{u}_2(t)| \leq 4c^2/\rho$ . The growth conditions mentioned in corollary 3.1 hold true for the game problem  $(\tilde{P})$ . Then from theorem 3.1 there exists a co-state function  $\tilde{\lambda}(t)$ , with Hamiltonian defined as

$$H(\cdot) := \tilde{\lambda}(t) (\tilde{u}_1(t) - \tilde{u}_2(t)) + (\alpha - (1 - \alpha)\tilde{x}^2(t)) (\tilde{u}_2(t) - \tilde{u}_1(t)).$$

Further,  $H(\cdot)$  attains a minimum w.r.t  $\tilde{u}_i(\cdot)$ ,  $i = 1, 2$  only if

$$\tilde{\lambda}(t) + (\alpha - (1 - \alpha)\tilde{x}^2(t)) = 0 \text{ for all } t \in [0, \infty). \quad (18)$$

As the growth conditions are satisfied we have  $\lim_{t \rightarrow \infty} \tilde{\lambda}(t) = 0$ . The adjoint variable  $\tilde{\lambda}(t)$  satisfies (by differentiating (18))

$$\dot{\tilde{\lambda}}(t) = 2(1 - \alpha)\tilde{x}^*(t)(\tilde{u}_2(t) - \tilde{u}_1(t)), \forall t \in [0, \infty) \quad \lim_{t \rightarrow \infty} \tilde{\lambda}(t) = 0.$$

We see that the necessary condition (7b) also results in the same differential equation for  $\tilde{\lambda}(t)$ ,  $\forall t \in [0, \infty)$ . Using (18) we can conclude that for arbitrary choices of  $u_1(\cdot)$  and  $u_2(\cdot)$ , theorem 3.1 holds true by choosing  $\alpha$  such that  $\lim_{t \rightarrow \infty} \tilde{x}^{*2}(t) = \frac{\alpha}{1-\alpha}$ . So, all the controls  $u_1(\cdot)$  and  $u_2(\cdot)$  are candidates for Pareto solutions (as they satisfy the necessary conditions). To show that the candidates are indeed Pareto optimal we have to show that the necessary conditions are sufficient too. This aspect is treated in example 3.

### 3.1 Discounted Autonomous systems

The growth conditions given in corollary 3.1 ensure that assumption 1 is satisfied. However, the conditions (14) are quite strict. In this subsection we analyze games defined by autonomous systems with exponentially discounted player's costs and for this class of problems assumption 1 is guaranteed under mild conditions. The discount factor  $\rho$  is assumed to be strictly positive. We represent the game problem as  $(P^\rho)$  and the related optimal control problem as  $(P_i^\rho)$ . For discounted autonomous systems, Mitchel [19] gives necessary conditions for optimality for free endpoint infinite horizon optimal control problems. However, in the present case the problems  $(P_i^\rho)$  are constrained with constraints taking a special structure. In the following discussion, owing to this special structure, we show that the conditions given by Mitchel [19] are sufficient to guarantee assumption 1. As a result, the necessary conditions for Pareto optimality of  $(P^\rho)$  are the same as the necessary conditions for optimality of a weighted sum optimal control problem.

As a first step, we derive the necessary conditions for optimality for the mixed endpoint constrained optimal control problem  $(P_i^\rho)$  (in similar lines of [19]<sup>3</sup>). If  $u^*$  is a Pareto optimal strategy for the game problem  $(P^\rho)$ , then from proposition 3.1  $u^*$  is optimal for the constrained optimal control problem  $(P_i^\rho)$ ,  $i \in \bar{N}$  given by:

$$\begin{aligned} (P_i^\rho) \quad & \min_{u \in \mathcal{U}} \int_0^\infty e^{-\rho t} g_i(x(t), u(t)) dt \\ \text{sub. to} \quad & \dot{x}(t) = f(x(t), u(t)), x(0) = x_0, u \in \mathcal{U} \\ & \dot{\tilde{x}}_j(t) = e^{-\rho t} g_j(x(t), u(t)), \tilde{x}_j^i(0) = 0, \lim_{t \rightarrow \infty} \tilde{x}_j^i(t) \leq \tilde{x}_j^*, \forall j \in \bar{N} \setminus \{i\}. \end{aligned}$$

Let  $(x^*(t), u^*(t))$ ,  $0 \leq t \leq \infty$  be the optimal admissible pair for problem  $(P_i^\rho)$ , we fix  $T > 0$  and define  $h_i(z)$ ,  $i \in \bar{N}$  as:

$$h_i(z) = \int_z^\infty e^{-\rho t} g_i(x^*(t - z + T), u^*(t - z + T)) dt. \quad (19)$$

<sup>3</sup>Notice, the necessary conditions given by Mitchel [19] considers only the free end point optimal control problem.

To derive the necessary conditions for optimality of  $u^*$ , we first consider the following truncated and augmented problem  $(P_{iT}^\rho)$  (associated with the problem  $(P_i^\rho)$ ) defined as follows:

$$\begin{aligned}
(P_{iT}^\rho) \quad & \min_{u \in \mathcal{U}} h_i(z(T) - T) + \int_0^T v(t) e^{-\rho z(t)} g_i(Y(t), U(t)) dt \\
\text{sub. to} \quad & \dot{Y}(t) = v(t) f(Y(t), U(t)), Y(0) = Y_0, Y(T) = x^*(T), U \in \mathcal{U} \\
& \dot{Y}_j^i(t) = v(t) e^{-\rho z(t)} g_j(Y(t), U(t)), \tilde{Y}_j^i(0) = 0, \tilde{Y}_j^i(T) + h_j(z(T) - T) \leq \tilde{x}_j^*, \forall j \in \bar{N} \setminus \{i\} \\
& \dot{z}(t) = v(t), z(0) = 0, v(t) \in [1/2, \infty).
\end{aligned} \tag{20}$$

*Remark 2.* Notice that the above problem is a mixed end point constrained finite horizon problem, i.e.,  $X(T)$  is fixed and  $Y_j^i(T)$  is constrained. Further, (20) captures the constraint set  $\mathcal{U}_i$  defined in lemma 2.2.

The following lemma, which is useful in theorem 3.2, relates the optimal solution of  $(P_i^\rho)$  to the optimal solution  $(P_{iT}^\rho)$ . The proof of lemma 3.1 is given in the appendix. We notice that the special structure of constraints given by (20) plays a role in arriving at this conclusion.

**Lemma 3.1.** *If  $(x^*(t), u^*(t))$  is an optimal admissible pair for the problem  $(P_i^\rho)$  then  $(x^*(t), t, u^*(t), 1)$ ,  $t \in [0, T]$ , is an optimal admissible pair for the problem  $(P_{iT}^\rho)$ .*

Using the above lemma, we give necessary conditions for optimality of problem  $(P_i^\rho)$  in the following theorem (see appendix for the proof).

**Theorem 3.2.** *If  $(x^*(t), u^*(t))$ ,  $t \in [0, \infty)$  is an optimal pair for the problem  $(P_i^\rho)$  then there exist  $\vec{\lambda}_i \in \mathbb{R}_+^N$ ,  $l_i^0 \in \mathbb{R}^n$  and continuous functions  $\lambda_i(t) \in \mathbb{R}^n$  and  $\gamma_i(t) \in \mathbb{R}$  respectively such that*

$$(\vec{\lambda}_i, \lambda_i(t), \gamma_i(t)) \neq (0, 0, 0), \forall t \geq 0, \left\| (\vec{\lambda}_i, \lambda_i(0)) \right\| = 1 \tag{21a}$$

$$\dot{\lambda}_i(t) = -e^{-\rho t} G_x(\vec{\lambda}_i, x^*(t), u^*(t)) - f'_x(x^*(t), u^*(t)) \lambda_i(t), \lambda_i(0) = l_i^0 \tag{21b}$$

$$\dot{\gamma}_i(t) = \rho e^{-\rho t} G(\vec{\lambda}_i, x^*(t), u^*(t)), \lim_{t \rightarrow \infty} \gamma_i(t) = 0 \tag{21c}$$

$$H(\vec{\lambda}_i, t, x^*(t), u(t), \lambda(t)) = \lambda_i'(t) f(x^*(t), u^*(t)) + e^{-\rho t} G(\vec{\lambda}_i, x^*(t), u^*(t))$$

$$H(\vec{\lambda}_i, t, x^*(t), u^*(t), \lambda(t)) \leq H(\vec{\lambda}_i, t, x^*(t), u(t), \lambda(t)) \quad \forall u(t) \tag{21d}$$

$$H(\vec{\lambda}_i, t, x^*(t), u^*(t), \lambda(t)) = -\gamma_i(t). \tag{21e}$$

*Remark 3.* Though the approach in lemma 3.1 and theorem 3.2 is similar to the one given in [19], the main differences lie in the problem formulation. In [19], the necessary conditions are obtained for the free endpoint unconstrained infinite horizon optimal control problem. However, the game problem  $(P^\rho)$ , due to proposition 3.1, leads to  $N$  mixed endpoint constrained optimal control problems  $(P_i^\rho)$ .

From theorem 3.2, if  $u^*$  is optimal for the problem  $(P_i)$  then there exists a  $\vec{\lambda}_i \in \mathbb{R}_+^N$  such that the conditions (21) hold true. These necessary conditions are closely related to the minimization of a weighted sum optimal control problem with the weight vector  $\vec{\lambda}_i$ . This observation is evident in the non-autonomous case as well, see (8) and (9). Due to the special structure of the constraint set  $\mathcal{U}_i$ , the term  $G(\vec{\lambda}_i, x^*(t), u^*(t))$ , weighted instantaneous undiscounted cost of the players, appears in the necessary conditions for optimality of all the problems  $(P_i^\rho)$ ,  $i \in \bar{N}$ . Now, from (21a) and (21c) if  $\lim_{t \rightarrow \infty} \lambda_i(t) = 0$  then assumption 1 holds true. As the scrap value associated with problem  $(P_i^\rho)$  is zero,  $\lim_{t \rightarrow \infty} \lambda_i(t) = 0$  is the natural transversality condition. In the discussion that follows we give two possible ways, in corollary 3.2 and corollary 3.3, to ensure  $\lim_{t \rightarrow \infty} \lambda_i(t) = 0$ . For autonomous systems [19] gives assumptions under which the natural transversality condition holds for the free endpoint case with maximization criterion. We show in the corollary 3.2 that these conditions, formulated as assumption 2 below, also suffice to conclude that this transversality condition holds for the problem  $(P_i^\rho)$ . The proof, given in the appendix, is along the same lines of [19], and requires the following assumption.

*Assumption 2.*  $g_i(x(t), u(t))$ ,  $\forall i \in \bar{N}$  is non positive and there exists a neighborhood  $V$  of  $0 \in \mathbb{R}^n$  which is contained in the set of possible velocities  $f(x^*(t), u(t))$  for all  $u \in \mathcal{U}$  if  $t \rightarrow \infty$ .

**Corollary 3.2.** *Let assumption 2 hold true. Then, an optimal solution for the problem  $(P_i^\rho)$  satisfies in addition to the conditions (21), the following transversality condition:  $\lim_{t \rightarrow \infty} \lambda_i(t) = 0$ .*

**Remark 4.** a) The nonpositivity assumption 2 can be relaxed. If the instantaneous undiscounted costs of players  $g_i(x(t), u(t))$ ,  $i \in \bar{N}$  are bounded above for all pairs  $(x(t), u(t))$ ,  $t \in [0, \infty)$  then by assigning a new cost  $\tilde{g}_i(x(t), u(t)) = g_i(x(t), u(t)) - M$  with  $M = \max_{i \in \bar{N}} \sup_{t \in [0, \infty)} g_i(x(t), u(t))$  leaves  $\tilde{g}(x(t), u(t))$  nonpositive. Now, by defining a new game  $(\tilde{P}^\rho)$  with  $\tilde{g}(\cdot)$  as the instantaneous undiscounted costs for player  $i$  we observe that Pareto optimal controls (if they exist) of  $(P^\rho)$  and  $(\tilde{P}^\rho)$  coincide. We will use this idea in example 2 to find Pareto optimal controls.

b) Notice that the second condition in assumption 2 is identical to the notion of state reachability when the state dynamics is described by a linear constant coefficient differential equation.

The conditions given in assumption 2 are mild but they are difficult to verify except for special cases, see remark 4(b) and example 2. Another possibility is to seek for (growth) conditions so as to obtain a bound on  $\|\lambda_i(t)\|$ . Recently, [23, 2] discuss such conditions for a class of free end point optimal control problems. In the following corollary 3.3, which is in the same spirit as the above works, we give (growth) conditions for the problem  $(P_i^\rho)$ . Towards that end, we make the following assumption. The proof of the corollary is provided in the appendix.

**Assumption 3.** a) There exist a  $s \geq 0$  and an  $r \geq 0$  such that

$$\|g_{ix}(x(t), u(t))\| \leq s(1 + \|x(t)\|^r) \text{ for all } x(t) \in \mathbb{R}^n, u \in \mathcal{U} \text{ and } i \in \bar{N}.$$

b) There exist nonnegative constants  $c_1, c_2, c_3$  and  $\lambda \in \mathbb{R}$ , such that for every admissible pair  $(x(t), u(t))$ , one has

$$\begin{aligned} \|x(t)\| &\leq c_1 + c_2 e^{\lambda t} \text{ for all } t \geq 0, \\ \|\Phi_{f_x}(t, 0)\| &\leq c_3 e^{\lambda t} \text{ for all } t \geq 0. \end{aligned}$$

c) For every admissible pair  $(x(t), u(t))$  the eigenvalues of  $f_x(x(t), u(t))$  are strictly positive.

**Corollary 3.3.** *Let assumption 3 hold true. Then, an optimal solution for problem  $(P_i^\rho)$  satisfies in addition to the conditions (21), the following transversality condition  $\lim_{t \rightarrow \infty} \lambda_i(t) = 0$  if  $\rho > (1 + r)\lambda$ .*

**Remark 5.** a) In the above corollary, by selecting a high discount factor, i.e., dominated discounting, the natural transversality condition is obtained. When the state evolution dynamics is linear and player's objectives are convex in the control variable, the growth conditions (a) and (b) given in assumption 3 are similar to those obtained in [2] and [3].

b) In [2], the free endpoint infinite horizon optimal control problem is approximated with a series of free endpoint finite horizon problems whereas in the current approach  $(P_i^\rho)$  is approximated with fixed endpoint problems. As a result, an additional condition (c) appears in the assumption 3, see the proof in appendix for more details.

To summarize, Corollaries 3.2 and 3.3 are sufficient conditions to ensure that assumption 1 holds true. Now, collecting the above results we proceed to the main result of the subsection.

**Theorem 3.3.** *Let assumption 2 or 3 hold true. If  $(J_1(u^*), J_2(u^*), \dots, J_N(u^*))$  is a Pareto optimal solution for problem  $(P)$  then there exists an  $\vec{\alpha} \in \mathcal{P}_N$  and a co-state function  $\lambda(t) : [0, \infty) \rightarrow \mathbb{R}^n$  such that the following conditions are satisfied.*

$$H(\vec{\alpha}, t, x(t), u(t), \lambda(t)) = \lambda'(t)f(t, x(t), u(t)) + e^{-\rho t}G(\vec{\alpha}, x(t), u(t)) \quad (22a)$$

$$H(\vec{\alpha}, t, x^*(t), u^*(t), \lambda(t)) \leq H(\vec{\alpha}, t, x^*(t), u(t), \lambda(t)), \forall u(t) \quad (22b)$$

$$H^0(\vec{\alpha}, t, x^*(t), \lambda(t)) = \min_{u(t) \in \mathcal{U}} H(\vec{\alpha}, t, x^*(t), u(t), \lambda(t))$$

$$\dot{\lambda}(t) = -H_x^0(\vec{\alpha}, t, x^*(t), \lambda(t)), \lambda(0) = l_0 \in \mathbb{R}^n, \lim_{t \rightarrow \infty} \lambda(t) = 0 \quad (22c)$$

$$\dot{x}^*(t) = H_x^0(\vec{\alpha}, t, x^*(t), \lambda(t)), x^*(0) = x_0 \quad (22d)$$

$$\dot{\gamma}(t) = \rho G(\vec{\alpha}, x^*(t), u^*(t)), \lim_{t \rightarrow \infty} \gamma(t) = 0 \quad (22e)$$

$$H^0(\vec{\alpha}, t, x^*(t), \lambda(t)) = -\gamma(t) \quad (22f)$$

$$(\vec{\alpha}, \gamma(t), \lambda(t)) \neq 0, \forall t \in [0, \infty), \vec{\alpha} \in \mathcal{P}_N. \quad (22g)$$

*Proof.* If assumption 2 or 3 holds true then for each problem  $(P_i^p)$   $\vec{\lambda}_i \in \mathbb{R}_+^N \setminus \{0\}$  and  $\lim_{t \rightarrow \infty} \lambda_i(t) = 0$ . We define  $d = \sum_{i \in \bar{N}} \vec{\lambda}_i' \mathbf{1}_N$ ,  $\alpha_i = \frac{1}{d} \left( \lambda_i^0 + \sum_{j \in \bar{N} \setminus \{i\}} \mu_i^j \right)$ ,  $i \in \bar{N}$  and a vector  $\vec{\alpha} = (\alpha_1, \dots, \alpha_N)'$ . We notice that  $\vec{\alpha} \in \mathcal{P}_N$ . Taking the summation of equation (21b) for all  $i \in \bar{N}$  and defining  $\lambda(t) = \frac{1}{d} \sum_{i \in \bar{N}} \lambda_i(t)$  we observe that conditions (22b) and (37) are satisfied. Taking the summation of equation (21c) for all  $i \in \bar{N}$  and defining  $\gamma(t) = \frac{1}{d} \sum_{i \in \bar{N}} \gamma_i(t)$ , the conditions (22e) and (22f) are satisfied. Since,  $\vec{\alpha} \in \mathcal{P}_N$  and  $\lim_{t \rightarrow \infty} \lambda(t) = 0$  we observe that (22g) is satisfied.  $\square$

Next, we consider an example from [14] to illustrate usage of assumption 2 and theorem 3.3.

*Example 2.* Consider a fishery game with two players. The evolution of the stock of fish, in a particular area, is governed by the differential equation

$$\dot{x}(t) = ax(t) - bx(t) \ln x(t) - u_1(t) - u_2(t), \quad x(0) = x_0 \geq 2 \quad (23)$$

where  $x(t)$  refers to the stock of fish, and  $a > 0$ ,  $b > 0$ . It is assumed that  $x(t) \geq 2, t \in [0, \infty)$ . In (23), the stock of fish  $x(t)$  depends upon  $ax(t)$  births,  $bx(t) \ln x(t)$  deaths and the fishing efforts of player  $i$ ,  $u_i(t) = w_i(t)x(t)$ , at each point in time  $t$ . Each fisherman tries to maximize his utility  $J_i(\cdot)$ , given by:

$$J_i(x_0, u_1, u_2) = \int_0^\infty e^{-\rho t} \ln u_i(t) dt.$$

We assume  $0 < \varepsilon \leq w_i(t) < \infty$  for the utility to be well defined. By taking the transformation  $y(t) = \ln x(t)$  the system (23) is modified as:

$$\dot{y}(t) = a - by(t) - w_1(t) - w_2(t), \quad y(0) = \ln x(0), \quad (24)$$

and the player's utility is transformed as:

$$J_i(x_0, u_1, u_2) = \int_0^\infty e^{-\rho t} (y(t) + \ln w_i(t)) dt. \quad (25)$$

We notice that the instantaneous undiscounted reward is bounded below. By controllability of the system (24) and remark 4.a we notice that assumption 2 is satisfied. So all Pareto candidates can be obtained by solving the necessary conditions associated with the weighted sum optimal control problem :

$$\min_{w_1, w_2} \left\{ - \int_0^\infty e^{-\rho t} (y(t) + \alpha \ln w_1(t) + (1 - \alpha) \ln w_2(t)) dt \right\},$$

subject to (24). Defining the Hamiltonian as

$$H(\alpha, t, y, w_1, w_2, \lambda) = \lambda (a - by(t) - w_1(t) - w_2(t)) - e^{-\rho t} (y(t) + \alpha \ln w_1(t) + (1 - \alpha) \ln w_2(t)).$$

Taking  $H_{w_i} = 0$ ,  $i = 1, 2$  gives  $w_1(t) = -\frac{\alpha}{\lambda(t)} e^{-\rho t}$  and  $w_2(t) = -\frac{1-\alpha}{\lambda(t)} e^{-\rho t}$ . The adjoint variable is governed by

$$\dot{\lambda}(t) = b\lambda(t) + e^{-\rho t}, \quad \lim_{t \rightarrow \infty} \lambda(t) = 0,$$

and the solution is given as  $\lambda(t) = -\frac{e^{-\rho t}}{\rho + b}$ . The candidates for Pareto optimal strategies in open loop form are given by:

$$\begin{aligned} u_1^*(t) &= \alpha(\rho + b)e^{m(t, x_0)}, \\ u_2^*(t) &= (1 - \alpha)(\rho + b)e^{m(t, x_0)}, \end{aligned}$$

where  $m(t, x_0) = e^{-bt} \ln x_0 + \frac{a - (b + \rho)}{b} (1 - e^{-bt})$ . The candidates for Pareto solutions are given as

$$\begin{aligned} J_1^*(x_0, u_1, u_2) &= \frac{\rho \ln x_0 + a - (\rho + b)}{\rho(\rho + b)} + \frac{\ln(\alpha(\rho + b))}{\rho}, \\ J_2^*(x_0, u_1, u_2) &= \frac{\rho \ln x_0 + a - (\rho + b)}{\rho(\rho + b)} + \frac{\ln((1 - \alpha)(\rho + b))}{\rho}. \end{aligned} \quad (26)$$

## 4 Sufficient Conditions for Pareto Optimality

It is well known [10] that if the action spaces as well as the players objective functions are convex then minimization of the weighted sum of the objectives results in all Pareto solutions. We give the following theorem from [7].

**Theorem 4.1.** *If  $\mathcal{U}$  is convex and  $J_i(u)$  is convex for all  $i = 1, 2, \dots, N$  then for all Pareto optimal  $u^*$  there exist  $\vec{\alpha} \in \mathcal{P}_N$ , such that  $u^* \in \arg \min_{u \in \mathcal{U}} \sum_{i=1}^N \alpha_i J_i(u)$ .*

Recently in [8], this property was used to obtain both necessary and sufficient conditions for the existence of Pareto optimal solutions for regular convex linear quadratic differential games. In general it is a difficult task to check if the players objectives are convex functions of initial state and controls. However, under some conditions the solutions of (29) result in Pareto optimal strategies. In this section we derive sufficient conditions for a strategy to be Pareto optimal. The sufficient conditions given in the theorem below are inspired by Arrow's sufficient conditions [23] in optimal control. Further, these sufficient conditions are given for non-autonomous systems and they hold true for discounted autonomous systems as well.

**Theorem 4.2.** *Assume that there exists  $\vec{\alpha} \in \mathcal{P}_N$  and a co-state function  $\lambda(t) : [0, \infty) \rightarrow \mathbb{R}^n$  satisfying (37). Introduce the Hamiltonian  $H(t, \vec{\alpha}, x(t), u(t), \lambda(t)) := f(t, x(t), u(t)) + G(\vec{\alpha}, t, x(t), u(t))$ . Assume that the Hamiltonian has a minimum w.r.t  $u(t)$  for all  $x(t)$ , denoted by*

$$H^0(\vec{\alpha}, t, x(t), \lambda(t)) = \min_{u(t)} H(\vec{\alpha}, t, x(t), u(t), \lambda(t)).$$

*If  $H^0(\vec{\alpha}, t, x(t), \lambda(t))$  is convex in  $x(t)$  and  $\liminf_{t \rightarrow \infty} \lambda'(t) (x^*(t) - x(t)) \geq 0$ , then  $u^*(t)$  is Pareto optimal.*

*Proof.* From the convexity of  $H^0(\vec{\alpha}, t, x(t), \lambda(t))$  we have:

$$H^0(\vec{\alpha}, t, x(t), \lambda(t)) - H^0(\vec{\alpha}, t, x^*(t), \lambda(t)) \geq H_x^0(\vec{\alpha}, t, x^*(t), \lambda(t)) (x(t) - x^*(t))$$

Since,  $H(\vec{\alpha}, t, x(t), u(t), \lambda(t)) \geq H^0(\vec{\alpha}, t, x(t), \lambda(t))$  and  $H(\vec{\alpha}, t, x^*(t), u^*(t), \lambda(t)) = H^0(\vec{\alpha}, t, x^*(t), \lambda(t))$  we have:

$$\begin{aligned} H(\vec{\alpha}, t, x(t), u(t), \lambda(t)) - H(\vec{\alpha}, t, x^*(t), u^*(t), \lambda(t)) &\geq H_x^{0'}(\vec{\alpha}, t, x^*(t), \lambda(t)) (x(t) - x^*(t)) \\ &= -\dot{\lambda}'(t) (x(t) - x^*(t)) \text{ (by (37)).} \end{aligned}$$

Using the definition of Hamiltonian the above inequality can be written as:

$$\lambda'(t) (f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))) + G(\vec{\alpha}, t, x(t), u(t)) - G(\vec{\alpha}, t, x^*(t), u^*(t)) \geq -\dot{\lambda}'(t) (x(t) - x^*(t)),$$

$$\begin{aligned} (G(\vec{\alpha}, t, x(t), u(t)) - G(\vec{\alpha}, t, x^*(t), u^*(t))) &\geq \dot{\lambda}'(t) (x^*(t) - x(t)) + \lambda'(t) (\dot{x}^*(t) - \dot{x}(t)) \\ &= \frac{d}{dt} (\lambda'(t) (x^*(t) - x(t))). \end{aligned}$$

Taking the integrals on both sides we have

$$\int_0^T (G(\vec{\alpha}, t, x(t), u(t)) - G(\vec{\alpha}, t, x^*(t), u^*(t))) dt \geq (\lambda'(t) (x^*(t) - x(t))) \Big|_0^T.$$

As  $x^*(0) = x(0) = x_0$  and  $\lambda(0)$  is bounded the above inequality is given as:

$$\int_0^T (G(\vec{\alpha}, t, x(t), u(t)) - G(\vec{\alpha}, t, x^*(t), u^*(t))) dt \geq (\lambda'(T) (x^*(T) - x(T))).$$

Taking  $T \rightarrow \infty$

$$J(u) - J(u^*) \geq \lim_{t \rightarrow \infty} \lambda'(t) (x^*(t) - x(t)) \geq \liminf_{t \rightarrow \infty} \lambda'(t) (x^*(t) - x(t)) \geq 0.$$

Clearly, by lemma 2.1  $u^*$  is Pareto optimal. □

**Example 3. (sufficient conditions):** We illustrate theorem 4.2 by considering example 1 again. First, we notice that  $H^0(t, \tilde{x}^*, \tilde{\lambda}) = 0$  so  $H^0(t, \tilde{x}^*, \tilde{\lambda})$  is convex in  $\tilde{x}(t)$ . Next,  $\lim_{t \rightarrow \infty} \tilde{x}(t)$  exists and is finite and  $\lim_{t \rightarrow \infty} \tilde{\lambda}(t) = 0$ , so  $\liminf_{t \rightarrow \infty} \tilde{\lambda}(t) (\tilde{x}^*(t) - \tilde{x}(t)) = 0$ . So, by theorem 4.2 every control  $(u_1, u_2)$  is Pareto optimal.

**Example 4. (sufficient conditions):** For example 2 the candidates for Pareto solutions are given by (26). If the model in example 2 satisfies the sufficient conditions mentioned in theorem 4.2, then all Pareto solutions are indeed given by (26). The minimized Hamiltonian is given by:

$$H^0(t, y(t), \lambda(t)) = -\frac{\rho y(t)}{\rho + b} e^{-\rho t} + e^{-\rho t} \left( 1 - \frac{a}{\rho + b} - \ln(\alpha^\alpha (1 - \alpha)^{1-\alpha} (\rho + b)) \right).$$

Clearly,  $H^0(t, y(t), \lambda(t))$  is convex (linear here) in  $y(t)$ . Since  $w_i(t)$ ,  $i = 1, 2$ , is bounded we have  $|y(t)| \leq (c_1 + c_2 e^{-bt})$ . Further,  $\lambda(t) = -\frac{e^{-\rho t}}{\rho + b}$ , thus we have  $\liminf_{t \rightarrow \infty} \lambda(t)(y^*(t) - y(t)) = 0$ . The fishery model satisfies the sufficient conditions as given by theorem 4.2. So, all the candidates given by (26) are Pareto solutions.

## 5 Linear Quadratic Case

In this section we consider the discounted infinite planning horizon linear quadratic cooperative differential game (denoted as  $(P^{\text{LQ}})$ ). Player  $i \in \bar{N}$  may choose his control trajectory,  $u_i(\cdot)$  from the set of admissible controls  $\mathcal{U}$  where the specific choice of control space will be clarified below. The problem is to determine the set of Pareto solutions for the cooperative game defined by

$$(P^{\text{LQ}}) \quad \min_{u(\cdot) \in \mathcal{U}} J_i(x_0, u) \quad (27)$$

$$\begin{aligned} J_i(x_0, u) &= \int_0^\infty e^{-\rho t} (x'(t) u'(t)) M_i (x'(t) u'(t))' dt, \\ \text{sub. to } \dot{x}(t) &= Ax(t) + \sum_{i \in \bar{N}} B_i u_i(t), \quad x(0) = x_0. \end{aligned} \quad (28)$$

We define the following spaces:

- a)  $L_{2,\text{loc}}^N := \{u \mid \int_0^\infty u'(t)u(t)dt < \infty\}$ , i.e., the set of locally square-integrable functions.
- b)  $L_{2,s}^+(x_0; x, A) := \left\{u \in L_{2,\text{loc}}^N \mid \text{s.t. } \lim_{t \rightarrow \infty} x(t) = 0, \dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0\right\}$ .

It can be proved, for instance see lemma 2.1 [9], that the control spaces mentioned above are convex. We take  $L_{2,\text{loc}}^N$  as the choice of control space  $\mathcal{U}$  unless otherwise specified. In the following corollary we give conditions under which assumption 1 is satisfied.

**Lemma 5.1.** *For  $(P^{\text{LQ}})$ , if the pair  $(A, B)$  is controllable then assumption 1 holds true.*

*Proof.* If  $u^*$  is Pareto optimal then by proposition 3.1  $u^*$  is optimal for the  $P_i^{\text{LQ}}$ ,  $i \in \bar{N}$ . The necessary conditions for optimality of  $u^*$  are given by equations (21). Assuming an interior solution  $u^*(t)$ , the first order condition translates to  $e^{-\rho t} (2R_{\lambda_i} u^*(t) + 2V_{\lambda_i} x^*(t)) + B' \lambda(t) = 0$ . If  $\vec{\lambda}_i = 0$ , then conditions lead to  $\dot{\lambda}_i(t) = -A' \lambda_i(t)$ ,  $\dot{\gamma}_i(t) = 0$  and the above first order condition would be  $B' \lambda_i(t) = 0$ . This implies  $B' \dot{\lambda}_i(t) = -B' A' \lambda_i(t) = 0$ . Repeating the same  $n - 1$  times we see that  $\lambda'(t) [B \quad AB \quad A^2 B \quad \dots \quad A^{n-1} B] = 0$ . As,  $(A, B)$  is controllable we necessarily have  $\lambda_i(t) = 0$  for all  $t$ . Further as  $\dot{\gamma}_i(t) = 0$  and  $\lim_{t \rightarrow \infty} \gamma_i(t) = 0$  we have  $\gamma_i(t) = 0$  for all  $t$ . But this violates the necessary condition (21a). So,  $\vec{\lambda}_i \in \mathbb{R}_+^N \setminus \{0\}$  and assumption 1 holds true.  $\square$

**Remark 6.** Specializing corollary 3.2 to the linear quadratic case to guarantee assumption 1 may require restrictions on the system parameters and control space  $\mathcal{U}^4$ .

In the next theorem we specialize theorem 3.3 for the linear quadratic case. Towards that end, we define  $z(t) = e^{-\rho t/2} x(t)$ ,  $v(t) = e^{-\rho t/2} u(t)$ ,  $p(t) = e^{\rho t/2} \lambda(t)$  and  $\tilde{A} = A - \frac{\rho}{2} I$ .

**Theorem 5.1.** *Let  $(A, B)$  be controllable. If  $(J_1(x_0, u^*), \dots, J_N(x_0, u^*))$  is a Pareto solution for the problem (27,28) then there exists an  $\vec{\alpha} \in \mathcal{P}_N$  such that*

$$\begin{aligned} e^{-\rho t} [x^{*'}(t) \quad u^{*'}(t)] M_\alpha [x^{*'}(t) \quad u^{*'}(t)]' + \lambda'(t) (Ax^*(t) + Bu^*(t)) \leq \\ e^{-\rho t} [x^{*'}(t) \quad u^{*'}(t)] M_\alpha [x^{*'}(t) \quad u'(t)]' + \lambda'(t) (Ax^*(t) + Bu(t)) \end{aligned} \quad (29a)$$

<sup>4</sup> A sufficient condition to satisfy assumption 1 can be shown as  $(A, B)$  controllable,  $A$  is stable and  $u(t) \in E \subset \mathbb{R}^m$ , with  $E$  being a bounded set.

$$\dot{x}^*(t) = Ax^*(t) + Bu^*(t), \quad x^*(0) = x_0, \quad (29b)$$

$$\dot{\lambda}(t) = -A'\lambda(t) - e^{-\rho t} (2Q_\alpha x^*(t) + 2S_\alpha u^*(t)), \quad \lambda(0) = l_0 \in \mathbb{R}^n \quad (29c)$$

In case  $\alpha$  is such that  $R_\alpha > 0$ , the above equations can be equivalently rephrased as that every Pareto optimal control satisfies  $u^*(t) = e^{\rho t/2} v^*(t)$  where  $v^*(t) = -R_\alpha^{-1} (S_\alpha z^*(t) + B' p(t))$ .  $(z^*(t), p(t))$  is the solution of the linear autonomous differential equation given by:

$$\begin{bmatrix} \dot{z}^*(t) \\ \dot{p}(t) \end{bmatrix} = G_\alpha \begin{bmatrix} z^*(t) \\ p(t) \end{bmatrix}, \quad z(0) = x_0 \text{ (given)}, \quad p(0) = l_0 \in \mathbb{R}^n. \quad (30)$$

*Proof.*  $(P^{\mathbb{L}\mathbb{Q}})$  is a special case of  $(PP)$ . Again using proposition 3.1 we have  $u^*$  is optimal for  $P_i^{\mathbb{L}\mathbb{Q}}$ ,  $i \in \bar{N}$ . Since  $(A, B)$  is controllable from lemma 5.1 assumption 1 holds true, i.e.,  $\vec{\lambda}_0^i \in \mathbb{R}_+^N \setminus \{0\}, i \in \bar{N}$ . So, the necessary conditions (29a-29c) follow directly from theorem 3.3.  $\square$

*Remark 7.* Here,  $G_\alpha$  is a Hamiltonian matrix given by

$$G_\alpha := \begin{pmatrix} \tilde{A} - BR_\alpha^{-1} S_\alpha & -BR_\alpha^{-1} B' \\ -(Q_\alpha - S_\alpha R_\alpha^{-1} S_\alpha) & -(\tilde{A} - BR_\alpha^{-1} S_\alpha)' \end{pmatrix}.$$

The extremal trajectories generated by the Hamiltonian flow (30) depend on  $x_0$  and  $\alpha$ . The additional information that we have is  $p(0) = l_0 \in \mathbb{R}^n$  is bounded. The eigenvalues of the Hamiltonian matrix  $G_\alpha$  are symmetric w.r.t real and imaginary axis. So,  $G_\alpha$  has at most  $n$  eigenvalues with negative real part. Bounded trajectories of (30) evolve on the stable manifolds and converge towards the equilibrium points of (30). The state of the Hamiltonian system,  $[z'(t) \ p'(t)]'$ , has  $2n$  variables and out of which only  $n$ , related to  $x_0$ , are free. The co-state variable  $p(t)$  can be obtained as a result of the above boundedness restriction and as a result depends on the initial state  $x_0$ . In nonlinear models, it is very common to have multiple co-state trajectories, converging to the equilibrium point, resulting in the same optimal cost, see [11, chapter 5].

To restrict the number of possible extremal trajectories we make the following assumption on admissible controls.

*Assumption 4.* The set of admissible controls  $v \in L_{2,s}^+(x_0, z, \tilde{A})$  where

$$L_{2,s}^+(x_0, z, \tilde{A}) := \left\{ v \in L_{2,\text{loc}}^N \mid \text{s.t. } \lim_{t \rightarrow \infty} z(t) = 0, \dot{z}(t) = \tilde{A}z(t) + Bv(t), z(0) = x_0 \right\}$$

Notice, assumption 4 requires  $\lim_{t \rightarrow \infty} z(t) = 0$  whereas  $x(t) = e^{\rho t/2} z(t)$  can grow unbounded. Strong restrictions on the system parameters ensure  $\lim_{t \rightarrow \infty} x(t) = 0$ , see section 5 [5] for more details. Theorem 5.1 only gives necessary conditions and solving these equations we obtain Pareto candidates. Further, we notice that these necessary conditions are similar, with controllability assumption, to necessary conditions for optimality of a weighted sum optimal control problem. The following theorem relates Pareto optimality with weighted sum minimization. We first define the weighted sum objective as:

$$J_\beta(x_0, u) := \int_0^\infty e^{-\rho t} [x'(t) \ u'(t)] M_\beta [x'(t) \ u'(t)]' dt$$

where  $\vec{\beta} \in \mathcal{P}_N$ ,  $R_i > 0, i \in \bar{N}$  and  $x(t)$  solves

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0. \quad (31)$$

**Theorem 5.2.** Let  $(A, B)$  be controllable. If  $u^*$  is Pareto optimal then there exists a  $\vec{\beta} \in \mathcal{P}_N$  such that the following condition holds true

$$J_\beta(x_0, u) - J_\beta(x_0, u^*) + \lim_{t \rightarrow \infty} \lambda'(t) (x(t) - x^*(t)) = J_\beta(0, u - u^*). \quad (32)$$

*Proof.* First we notice,

$$\begin{aligned} J_\beta(x_0, u) - J_\beta(x_0, u^*) &= \int_0^\infty e^{-\rho t} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' M_\beta \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt - \int_0^\infty e^{-\rho t} \begin{bmatrix} x^*(t) \\ u^*(t) \end{bmatrix}' M_\beta \begin{bmatrix} x^*(t) \\ u^*(t) \end{bmatrix} dt \\ &= \int_0^\infty e^{-\rho t} \begin{bmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{bmatrix}' M_\beta \begin{bmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{bmatrix} dt + 2 \int_0^\infty e^{-\rho t} \begin{bmatrix} x^*(t) \\ u^*(t) \end{bmatrix}' M_\beta \begin{bmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{bmatrix} dt \end{aligned}$$

We recognize the first part of the sum on the righthand side of the above equations as  $J_\beta(0, u - u^*)$ . Since  $u^*$  is Pareto optimal from theorem 5.1 there exists a  $\vec{\beta} \in \mathcal{P}_N$  such that (29a-29c) hold true. Now, we observe using conditions (29a) and (29c) that

$$\begin{aligned} \lim_{t \rightarrow \infty} \lambda'(t) (x(t) - x^*(t)) - \lambda'(0) (x(0) - x^*(0)) &= \int_0^\infty \frac{d}{dt} (\lambda'(t) (x(t) - x^*(t))) dt \\ &= -2 \int_0^\infty e^{-\rho t} \left[ \begin{matrix} x^*(t) \\ u^*(t) \end{matrix} \right]' M_\beta \left[ \begin{matrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{matrix} \right] dt \end{aligned}$$

Since  $\lambda(0)$  is bounded, the second term in sum on the lefthand side of the above equation vanishes and results in equation (32).  $\square$

Theorem 5.3 given below states that, under controllability condition, for a fixed initial state a weighted sum (single player) linear quadratic optimal control problem has a solution if and only if the cost function is convex in  $u$ , the necessary conditions resulting from the maximum principle and a transversality condition are satisfied.

**Theorem 5.3.** *We have the following assertions for  $J_\beta(x_0, v)$ .*

a) (Convexity) For any  $\alpha \in [0, 1]$ ,  $u_i \in \mathcal{U}$ ,  $i = 1, 2$  and  $\vec{\beta} \in \mathcal{P}_N$  we have

$$\alpha J_\beta(x_0, u_1) + (1 - \alpha) J_\beta(x_0, u_2) - J_\beta(x_0, \alpha u_1 + (1 - \alpha) u_2) = \alpha(1 - \alpha) J_\beta(0, u_1 - u_2). \quad (33)$$

b) Let  $(A, B)$  be controllable, then  $u^* = \arg \min_{u \in \mathcal{U}} J_\beta(x_0, u)$  exists if

i)  $\min_{u \in \mathcal{U}} J_\beta(0, u)$  exists (and equals zero).

ii) there exist  $u^*$ ,  $x^*$  and  $\lambda$  that satisfy, for all  $t \geq 0$ ,

$$e^{-\rho t} (2Q_\beta x^*(t) + 2R_\beta u^*(t)) + B' \lambda(t) = 0, \text{ where} \quad (34a)$$

$$\dot{x}^*(t) = Ax^*(t) + Bu^*(t), \quad x^*(0) = x_0, \quad (34b)$$

$$\dot{\lambda}(t) = -A' \lambda(t) - e^{-\rho t} (2Q_\beta x^*(t) + 2S_\beta u^*(t)), \quad \lambda(0) = l_0 \in \mathbb{R}^n. \quad (34c)$$

Conversely, if (i), (ii) and in addition  $\liminf_{t \rightarrow \infty} \lambda'(t) (x^*(t) - x(t)) \geq 0$  holds true then  $u^* = \arg \min_{u \in \mathcal{U}} J_\beta(x_0, u)$ .

*Proof.* a) By linearity of the system (31) if  $x_i(t)$  is generated by  $u_i(t)$  with  $x_i(0) = x_0$  for  $i = 1, 2$ . Then for  $\alpha \in [0, 1]$ ,  $\alpha u_1(t) + (1 - \alpha) u_2(t)$  generates  $\alpha x_1(t) + (1 - \alpha) x_2(t)$  with initial state as  $x_0$ .

$$\begin{aligned} J_\beta(x_0, \alpha u_1(t) + (1 - \alpha) u_2(t)) &= \int_0^\infty e^{-\rho t} \left[ \begin{matrix} \alpha x_1(t) + (1 - \alpha) x_2(t) \\ \alpha u_1(t) + (1 - \alpha) u_2(t) \end{matrix} \right]' M_\beta \left[ \begin{matrix} \alpha x_1(t) + (1 - \alpha) x_2(t) \\ \alpha u_1(t) + (1 - \alpha) u_2(t) \end{matrix} \right] dt \\ &= \alpha^2 J_\beta(x_0, u_1) + (1 - \alpha)^2 J_\beta(x_0, u_2) + 2\alpha(1 - \alpha) \int_0^\infty e^{-\rho t} \left[ \begin{matrix} x_1(t) \\ u_1(t) \end{matrix} \right]' M_\beta \left[ \begin{matrix} x_2(t) \\ u_2(t) \end{matrix} \right] dt \end{aligned}$$

$$\begin{aligned} \alpha J_\beta(x_0, u_1) + (1 - \alpha) J_\beta(x_0, u_2) - J_\beta(x_0, \alpha u_1(t) + (1 - \alpha) u_2(t)) &= \\ \alpha(1 - \alpha) \int_0^\infty e^{-\rho t} \left[ \begin{matrix} x_1(t) - x_2(t) \\ u_1(t) - u_2(t) \end{matrix} \right]' M_\beta \left[ \begin{matrix} x_1(t) - x_2(t) \\ u_1(t) - u_2(t) \end{matrix} \right] dt \end{aligned}$$

Again using the linearity property we identify the integral on the right hand side as  $J_\beta(0, u_1 - u_2)$ .

b)  $\Rightarrow$  First, we have  $x(t, x_0, u) = e^{At} x_0 + \int_0^t e^{A(t-s)} Bu(s) ds$ . So, introducing  $p(t) = e^{At} x_0$  and  $w(t) = \int_0^t e^{A(t-s)} Bu(s) ds$ , we have  $x(t) = p(t) + w(t)$ . Some elementary calculations then show that

$$J(x_0, u) = J(0, u) + \int_0^\infty e^{-\rho s} p'(s) Q_0 p(s) ds + 2 \int_0^\infty e^{-\rho s} (p'(s) Q_0 w(s) + p'(s) V_0 u(s)) ds$$

Therefore for any  $\tau \in \mathbb{R}$  we have

$$J(x_0, \tau u) = \tau^2 J(0, u) + \int_0^\infty e^{-\rho s} p'(s) Q_0 p(s) ds + 2\tau \int_0^\infty e^{-\rho s} (p'(s) Q_0 w(s) + p'(s) V_0 u(s)) ds$$



So, if  $J(0, u) < 0$  for some  $u \in \mathcal{U}$ ,  $J(z_0, \tau u)$  can be made arbitrarily small by choosing  $\tau$  large enough. Therefore we conclude that if  $\min_{u \in \mathcal{U}} J(z_0, u)$  exists, necessarily  $J(0, u) \geq 0$  for every  $u \in \mathcal{U}$ . Since  $J(0, 0) = 0$  it is obvious that condition (i) holds. Since  $u^*$  is a minimizer the necessary conditions for optimality hold in normal form following the reasoning given by lemma 5.1, and (ii) holds true.

$\Leftarrow$  If there exists a  $u^*$  satisfying (34) then following the proof of theorem 5.2 we have  $J_\beta(x_0, u) - J_\beta(x_0, u^*) + \lim_{t \rightarrow \infty} \lambda'(t)(x(t) - x^*(t)) = J_\beta(0, u - u^*)$  for all  $u \in \mathcal{U}$ . From (i) we have  $J_\beta(x_0, u) - J_\beta(x_0, u^*) \geq \lim_{t \rightarrow \infty} \lambda'(t)(x^*(t) - x(t))$ . By assumption, we have  $\liminf_{t \rightarrow \infty} \lambda'(t)(x^*(t) - x(t)) \geq 0$ . So,  $J_\beta(x_0, u) \geq J_\beta(x_0, u^*)$  for all  $u \in \mathcal{U}$ .  $\square$

## 5.1 Fixed Initial State

In this section we give additional properties of Pareto solutions that arise due to linear quadratic nature of the game ( $P^{\text{LQ}}$ ). First, the following properties hold true due to the linearity of the game ( $P^{\text{LQ}}$ ), refer lemma 3.2 [9] for a detailed proof.

**Lemma 5.2.** *Assume  $u^*$  is a Pareto optimal control for (27, 28). Then  $\mu u^*$  is a Pareto optimal control for (27, 28) with  $x(0) = \mu x_0$ .*

As a result of the above lemma, if for the initial state  $x_0 = 0$  there exists a Pareto solution different from zero, then all points on the half-line connecting this point and zero are also Pareto solutions. We state this observation formally in the next corollary.

**Corollary 5.1.** *Consider the cooperative game (27, 28) with  $x_0 = 0$ . Assume  $u^*$  is a Pareto optimal control for this game yielding the Pareto solution  $J^*$ . Then for all  $\mu \in \mathbb{R}$ ,  $\mu u^*$  yields the Pareto solution  $\mu^2 J^*$ .*

For the two player case in a finite horizon setting theorem 3.6 [9] shows that all Pareto solutions can be obtained using the weighting method using a technical lemma 3.5 [9]. The following theorem is an infinite horizon counter part of theorem 3.6 [9].

**Theorem 5.4.** *Let  $(A, B)$  be controllable. Consider the two-player case of the problem with  $R_i > 0$ .*

1. *If  $(J_1(x_0, u^*), J_2(x_0, u^*))$  is a Pareto solution for problem ( $P^{\text{LQ}}$ ), then there exists an  $\alpha \in [0, 1]$  such that*
  - (i) *(34) hold (with  $u^*$  defined correspondingly) and*
  - (ii) *for this  $\alpha$ ,  $\inf_u J_\alpha(0, u) := \inf_u (\alpha J_1(0, u) + (1 - \alpha) J_2(0, u))$  exists.*
2. *Conversely, if there exists an  $\alpha \in (0, 1)$  such that (i) and (ii) above hold true, then  $(J_1(x_0, u^*), J_2(x_0, u^*))$  is a Pareto solution.*

*Proof.* 1) Following lemma 3.5 [9] and theorem 3.6, item (1) [9] if  $(J_1(x_0, u^*), J_2(x_0, u^*))$  is a Pareto solution then  $u^*$  minimizes the weighted sum  $J_\alpha(x_0, u)$ . The remaining part follows from theorem 5.3 (b). 2) Follows from theorem 5.3 (b) and direct application of lemma 2.1.  $\square$

**Remark 8.** Note that theorem 5.4 does not assume that the cost functions  $J_i$  are convex. Further, almost all Pareto solutions can be obtained using the weighting method. The only additional Pareto optimal solutions that may exist are obtained by considering strategies  $\tilde{u} = \arg \min_{u \in \mathcal{U}} J_i(x_0, u)$  for some  $i \in \overline{N}$ . It is still unclear if the same conclusion can be derived in a  $N(> 2)$  player setting.

## 5.2 Arbitrary Initial State

In this section we consider conditions under which ( $P^{\text{LQ}}$ ) has a Pareto solution for an arbitrary initial state. It is well known [28, 30] that the solution of (30) is closely related to the existence of the solution to the following algebraic Riccati equation (ARE):

$$\tilde{A}'X + X\tilde{A} - (XB + S_\alpha)R_\alpha^{-1}(B'X + S_\alpha) + Q_\alpha = 0 \quad (\text{ARE}).$$

Using theorem 5 [28] and theorem 13.9 [30], we state the following proposition:

**Proposition 5.1.** *Let  $(\tilde{A}, B)$  be controllable. Then the following are equivalent.*

a) The frequency domain inequality (FDI) satisfies

$$\Psi_\alpha(j\omega) = \begin{bmatrix} B'(-j\omega - \tilde{A}')^{-1} & I \end{bmatrix} M_\alpha \begin{bmatrix} (j\omega - \tilde{A})^{-1} B \\ I \end{bmatrix} \geq \varepsilon B'(-j\omega - \tilde{A}')^{-1} (j\omega - \tilde{A})^{-1} B$$

for some  $\varepsilon > 0$  and  $0 \leq \omega \leq \infty$ .

b) The (ARE) has a unique real symmetric stabilizing solution  $X_\alpha$  such that  $\sigma(\tilde{A} - BR_\alpha^{-1}(B'X_\alpha + S_\alpha)) \in \mathbb{C}_-$ .

c) The Hamiltonian matrix  $G_\alpha$  has no  $j\omega$ -axis eigenvalues and there exists an  $n$  dimensional stable graph subspace<sup>5</sup>.

**Lemma 5.3.** Let  $(\tilde{A}, B)$  be controllable. If the (ARE) has a real symmetric stabilizing solution for  $\vec{\alpha}_1, \dots, \vec{\alpha}_k \in \mathcal{P}_N$  then (ARE) has a solution for all  $\vec{\alpha}$  in the cone  $\mathcal{K}(\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_k)$ , where

$$\mathcal{K}(\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_k) := \left\{ \vec{\alpha} \in \mathcal{P}_N \mid \vec{\alpha} = \sum_{i=1}^k \kappa_i \vec{\alpha}_i, \kappa_i > 0, i = 1, 2, \dots, k \right\}.$$

*Proof.* From proposition 5.1(b), if the (ARE) has a solution for  $\vec{\alpha}_i \in \mathcal{P}_N$  then  $\Psi_{\alpha_i}(j\omega)$  satisfies the inequality given in 5.1(a) for  $i = 1, 2, \dots, k$ . So, for any  $\vec{\alpha} \in \mathcal{K}(\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_k)$ ,  $\Psi_\alpha(j\omega) = \sum_{i=1}^k \kappa_i \Psi_{\alpha_i}(j\omega) \geq \varepsilon B'(-j\omega - \tilde{A}')^{-1} (j\omega - \tilde{A})^{-1} B$  for some  $\varepsilon > 0$  and  $0 \leq \omega \leq \infty$ . Again from proposition 5.1 we have that (ARE) with  $\vec{\alpha} \in \mathcal{K}(\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_k)$  has a real symmetric stabilizing solution.  $\square$

In the theorem 5.5 given below we consider the special case when (ARE) has a stabilizing solution for the vertices of the simplex  $\mathcal{P}_N$ .

**Theorem 5.5.** Let  $(\tilde{A}, B)$  be stabilizable and  $v \in L_{2,s}^+(x_0, z, \tilde{A})$ . Assume (ARE) has a solution for  $\vec{\alpha} = \vec{e}_i$ ,  $i = 1, 2, \dots, N$ , where  $\vec{e}_i$  is the  $i^{\text{th}}$  standard unit vector in  $\mathbb{R}^N$ . Then for all initial states a Pareto solution exists. For a fixed initial state the set of all Pareto solutions is given by  $\{(J_1(u_\alpha^*), \dots, J_N(u_\alpha^*))\}$ . Here, for a fixed  $\vec{\alpha} \in \mathcal{P}_N$ ,

$$u_\alpha^*(t) = -e^{-\rho t/2} R_\alpha^{-1} (B'X_\alpha + V_\alpha) z^*(t)$$

where  $z^*(t)$  satisfies  $\dot{z}^*(t) = (\tilde{A} - BR_\alpha^{-1}(B'X_\alpha + V_\alpha)) z^*(t)$ ,  $z^*(0) = x_0$ .

*Proof.* If  $\vec{\alpha} = \vec{e}_i$  then the game problem reduces to a single player optimal control problem. Recalling theorem 2.7 [9], if  $(\tilde{A}, B)$  is stabilizable then  $\min_{u \in \mathcal{U}} J_i(x_0, u)$  exists if and only if (ARE) has a unique stabilizing real symmetric solution  $X$ . Under this condition  $\min_{u \in \mathcal{U}} J_i(x_0, u) = x_0' X_i x_0$  is attained uniquely by  $u^*(t) = -e^{\rho t/2} R_i^{-1} (B'X_i + V_i) z^*(t)$ , where  $z^*(\cdot)$  solves  $\dot{z}^*(t) = (\tilde{A} - BR_i^{-1}(B'X_i + V_i)) z^*(t)$ . Clearly, we have  $J_i(0, u) \geq \min_u J_i(0, u) = 0$ . So, from theorem 5.3,  $J_i(x_0, u)$  is strictly convex in  $u$ . As a result if (ARE) has a real symmetric stabilizing solution for  $\vec{\alpha} = \vec{e}_i$ ,  $i = 1, 2, \dots, N$  then players' objectives  $J_i(x_0, u)$  are strictly convex in  $u$ . Since the choice of control space is convex, from theorem 4.1 it follows that for all Pareto optimal  $u^*$  there exist  $\vec{\alpha} \in \mathcal{P}_N$  such that  $u^* := \arg \min_{u \in \mathcal{U}} \sum_{i=1}^N \alpha_i J_i(x_0, u)$  for all  $x_0$ . Notice, we only require that  $(\tilde{A}, B)$  to be stabilizable for this conclusion. Further, there exists a one-one correspondence between Pareto surface and  $\mathcal{P}_N$ .  $\square$

In the following lemma we show under certain conditions that a Pareto optimal control minimizes a weighted sum optimal control problem.

**Lemma 5.4.** Let  $(\tilde{A}, B)$  be controllable and  $v \in L_{2,s}^+(x_0, z, \tilde{A})$ . If  $u^*$  is Pareto optimal then there exists a  $\vec{\beta} \in \mathcal{P}_N$  such that conditions (32) holds true. Further, if (ARE) has a unique real symmetric stabilizing solution for this  $\vec{\beta}$  then  $u^*$  minimizes  $J_\beta(x_0, u)$ .

*Proof.* If  $u^*$  is Pareto optimal then from theorem 5.2 there exists a  $\vec{\beta} \in \mathcal{P}_N$  such that (32) holds true. Suppose if (ARE) has a unique real symmetric stabilizing solution  $X_\beta$  then  $J_\beta(x_0, u)$  is strictly convex, and as a result there exists a unique  $\tilde{u} = \arg \min_{u \in \mathcal{U}} J_\beta(x_0, u)$  such that  $J(x_0, \tilde{u}) = x_0' X_\beta x_0$ , in particular  $\min_{u \in \mathcal{U}} J_\beta(0, u) = 0$ . To show  $u^* = \tilde{u}$  we proceed as follows. From (32) and above arguments we have  $J_\beta(x_0, u) - J_\beta(x_0, u^*) + \lim_{t \rightarrow \infty} \lambda'(x(t) - x^*(t)) \geq 0$ . With

<sup>5</sup>Let  $S$  represent the subspace spanned by eigen vectors, denoted by  $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_{2n \times n}$ , associated with stable eigen values. If  $X_{1n \times n}$  is invertible then we call  $S$  a stable graph subspace.

straightforward calculations we can show that  $\lim_{t \rightarrow \infty} p'(t)(z^*(t) - z(t)) = \lim_{t \rightarrow \infty} \lambda'(t)(x^*(t) - x(t))$ . From, proposition 5.1 we have that  $G_\beta$  has an  $n$  dimensional stable graph subspace. So, the optimal co-state rule is given uniquely by  $p(t) = X_\beta z^*(t)$ . As  $X_\beta$  is stabilizing and  $v \in L_{2,s}^+(x_0, z, \tilde{A})$ , we have that  $\lim_{t \rightarrow \infty} (z(t) - z^*(t)) = 0$ , and as a result  $\lim_{t \rightarrow \infty} p'(t)(z(t) - z^*(t)) = 0$ . Clearly,  $J_\beta(x_0, u) - J_\beta(x_0, u^*) \geq 0$  for all  $x_0$ . So,  $u^*$  also minimizes  $J_\beta(x_0, u)$ . From the uniqueness of the minimizer we have  $u^* = \tilde{u}$ .  $\square$

A question which next naturally arises is whether we can characterize all Pareto solutions in a way similar to theorem 5.5 if there exists some player who can obtain arbitrarily low costs if he is allowed to manipulate all control instruments that affect the system, that is, if not all cost functions are convex. Such situations occur if, say player 1 could, by choosing the actions of player 2, achieve arbitrarily low costs (i.e., gains). This occurs at the expense of player 2, whose costs increase using the corresponding control scheme. We use lemma 5.3 and lemma 5.4 to address this issue in the following corollary.

**Corollary 5.2.** *Let  $(A, B)$  be controllable and  $v \in L_{2,s}^+(x_0, z, \tilde{A})$ . Consider the problem with  $R_i > 0$ . Assume (ARE) has a solution for  $\vec{\alpha}_i \in \mathcal{P}_N$ ,  $i = 1, 2, \dots, k$ . Then for all initial states a Pareto solution exists. For a fixed initial state and for all  $\vec{\alpha} \in \mathcal{H}(\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_k)$ ,  $\{(J_1(u_\alpha^*), \dots, J_N(u_\alpha^*))\}$  yield Pareto solutions. Here  $u_\alpha^*(t) := e^{-\rho t/2} v^*(t)$ ,  $v^*(t) = -R_\alpha^{-1}(V_\alpha + B'X_\alpha)z^*(t)$ ,  $z^*(t)$  solves the differential equation  $\dot{z}^*(t) = (\tilde{A} - BR_\alpha^{-1}(V_\alpha + B'X_\alpha))z^*(t)$ ,  $z^*(0) = x_0$  and  $X_\alpha$  solves (ARE).*

The number of extremal trajectories are considerably reduced by assumption 4. The co-state rule which defines the co-state trajectory, in general, depends on the initial state. However, if (ARE) has a unique stabilizing solution it is defined by  $p(t) = X_\alpha z^*(t)$  and doesnot depend on the choice of the initial state. The following example demonstrates these subtleties.

*Example 5.* Consider the cooperative game with

$$\begin{aligned} J_i(x_0, u) &= \int_0^\infty e^{-\rho t} \sum_{j=1}^2 (q_{ij}^2 x_j^2(t) + r_{ij}^2 u_j^2(t)) dt, \quad i = 1, 2 \\ \text{sub. to} \quad \dot{x}(t) &= (A + \frac{\rho}{2}I)x(t) + B_1 u_1(t) + B_2 u_2(t), \quad x(0) = x_0 \neq 0, \quad \rho > 0 \\ v(t) &= e^{\rho t/2} v(t), \quad v \in L_{2,s}^+(x_0, z, \tilde{A}) \end{aligned}$$

Here we take  $A = \text{diag}(a_1, a_2)$ ,  $[B_1 \ B_2] = I$ . Choosing  $r_{ij} > 0$ ,  $q_{i2} > 0$ ,  $i, j = 1, 2$ ,  $q_{11} = -a_1^2 r_{11}$  and  $q_{21} = -a_1^2 r_{21}$  we can show that the eigenvalues of the Hamiltonian matrix  $G_\alpha$  as  $\{-s, 0, 0, s\}$ , where  $s = \sqrt{a_2^2 + \frac{\alpha q_{12} + (1-\alpha)q_{22}}{\alpha r_{12} + (1-\alpha)r_{22}}} > 0$ . The eigenvector and generalized eigenvector corresponding to eigenvalue zero are  $v_0^1 = [1 \ 0 \ a_1(\alpha r_{11} + (1-\alpha)r_{12}) \ 0]'$  and  $v_0^2 = [1/a_1 \ 0 \ 0 \ 0]'$ . The eigenvector corresponding to the stable eigenvalue  $-s$  is calculated as  $v_{-s}^1 = [0 \ \frac{s-a_2}{\alpha q_{12} + (1-\alpha)q_{22}} \ 0 \ 1]'$ . The admissible extremal trajectories could belong to one of the subspaces, namely  $\text{sp}\{v_0^1\}$ ,  $\text{sp}\{v_0^1, v_0^2\}$ ,  $\text{sp}\{v_{-s}^1\}$  and  $\text{sp}\{v_{-s}^1, v_0^1\}$ . Taking assumption 4 some trajectories can be ruled out.

$(z^*(t), p(t)) \in \text{sp}\{v_0^1\}$ : In this case the we obtain  $v_1^*(t) = -a_1 z_1^*(t)$ ,  $v_2^*(t) = 0$ ,  $z_1^*(t) = x_1(0)$ ,  $z_2^*(t) = e^{a_2 t} x_2(0) = 0$ ,  $p_1(t) = -a_1(\alpha r_{11} + (1-\alpha)r_{12})z_1^*(t)$  and  $p_2(t) = 0$  for all  $t$ .

$(z^*(t), p(t)) \in \text{sp}\{v_0^1, v_0^2\}$ : In this case the we obtain, in addition to the above extremal,  $v_1^*(t) = 0$ ,  $v_2^*(t) = 0$ ,  $z_1^*(t) = e^{a_1 t} x_1(0)$ ,  $z_2^*(t) = e^{a_2 t} x_2(0) = 0$ ,  $p_1(t) = 0$  and  $p_2(t) = 0$  for all  $t$ .

$(z^*(t), p(t)) \in \text{sp}\{v_{-s}^1, v_{-s}^1\}$ : In this case the we obtain  $v_1^*(t) = -a_1 z_1^*(t)$ ,  $v_2^*(t) = -\frac{(s-a_2)(\alpha r_{12} + (1-\alpha)r_{22})}{\alpha q_{12} + (1-\alpha)q_{22}} z_2^*(t)$ ,  $z_1^*(t) = x(0)$ ,  $z_2^*(t) = e^{-st} x_2(0)$ ,  $p_1(t) = a_1(\alpha r_{11} + (1-\alpha)r_{12})z_1^*(t)$  and  $p_2(t) = \frac{(s-a_2)}{\alpha q_{12} + (1-\alpha)q_{22}} z_2^*(t)$  for all  $t$ .

$(z^*(t), p(t)) \in \text{sp}\{v_{-s}^1\}$ : In this case the we obtain  $v_1^*(t) = 0$ ,  $v_2^*(t) = -\frac{(s-a_2)(\alpha r_{12} + (1-\alpha)r_{22})}{\alpha q_{12} + (1-\alpha)q_{22}} z_2^*(t)$ ,  $z_1^*(t) = e^{a_1 t} x(0) = 0$ ,  $z_2^*(t) = e^{-st} x_2(0)$ ,  $p_1(t) = 0$  and  $p_2(t) = \frac{(s-a_2)}{\alpha q_{12} + (1-\alpha)q_{22}} z_2^*(t)$  for all  $t$ .

Since  $v \in L_{2,s}^+(x_0, z, \tilde{A})$ , admissible extremals satisfy  $\lim_{t \rightarrow \infty} z^*(t) = 0$ . In the first two cases the obtained extremal trajectories are possible only with  $x_0 = 0$ . In the latter two cases we see that the obtained extremals are possible with  $x_0 = (0, x_2(0))$  where  $x_2(0)$  is arbitrary. Further, these extremals are the same. Now we check if this extremal is Pareto optimal using the sufficient conditions given by theorem 4.2. The minimized Hamiltonian is given by  $H^0(\cdot) := 0$ .

Since admissibility requires  $\lim_{t \rightarrow \infty} z(t) = 0$  we see that  $\lim_{t \rightarrow \infty} p'(t)(z^*(t) - z(t)) = 0$ . So, the obtained controls are indeed Pareto optimal. Further, we see that not all initial states, in particular  $x_1(0) \neq 0$ , result in a Pareto solution.

### 5.3 The scalar case

In this subsection we discuss the scalar case in more detail. Notice, assuming  $b_i \neq 0$ ,  $i = 1, 2, \dots, N$ , controllability is trivially satisfied. Using lemma 5.2 some interesting observations can be derived for the scalar case. The choice of control space is taken as  $L_{2,s}^+(x_0, z, \tilde{A})$ . So, all the admissible trajectories satisfy  $\lim_{t \rightarrow \infty} z(t) = 0$ . The Hamiltonian

matrix takes the form  $G_\alpha = \begin{bmatrix} e & -f \\ -g & -e \end{bmatrix}$ , with  $f > 0$ . Now, we have the following 3 possible cases.

- a)  $G_\alpha$  has eigenvalue zero with geometric multiplicity 2. Straightforward calculations show that  $z(t) = x_0 + (ex_0 - fl_0)t$  and  $p(t) = l_0 - (gx_0 - el_0)t$ . For admissibility of extremal trajectories we require  $\lim_{t \rightarrow \infty} z(t) = 0$ , and this is possible only if  $x_0 = 0$  and  $l_0 = 0$  as  $f \neq 0$ .
- b)  $G_\alpha$  has complex eigenvalues. Again, straight forward calculations show that  $z(t) = x_0 \cos(wt) + (ex_0 - fl_0) \sin(wt)$  and  $p(t) = l_0 \cos(wt) - (gx_0 + el_0) \sin(wt)$ . Following the same reasoning as above we have that  $x_0 = 0$  and  $l_0 = 0$ .
- c) The eigenvector corresponding to stable eigenvalue is always a graph subspace. If  $\sigma > 0$  is an eigenvalue of  $G_\alpha$  then  $-\sigma$  is also an eigenvalue of  $G_\alpha$ . As  $f > 0$ , the eigenvector corresponding to  $-\sigma$  can always be taken as  $[1 \quad (e + \sigma)/f]^T$ .

Theorem 5.6 below states that if  $(P^{\mathbb{LQ}})$  has a Pareto solution for a non zero initial state, then it can be found using the weighting method.

**Lemma 5.5.** *Consider the scalar system with  $R_i > 0$ ,  $i \in \bar{N}$  and  $v \in L_{2,s}^+(x_0, z, \tilde{A})$ . Let  $x_0 \neq 0$ . If  $(P^{\mathbb{LQ}})$  has a Pareto optimal control  $\tilde{u}^*(x_0)$  then there exists an  $\vec{\alpha} \in \mathcal{P}_N$  such that  $\min_{\sum_{i \in \bar{N}} \alpha_i J_i}$  subject to (28) has a solution for all initial states (including  $x_0 = 0$ ).*

*Proof.* Let  $(J_1(x_0, u^*), J_2(x_0, u^*), \dots, J_N(x_0, u^*))$  is a Pareto solution for some  $x_0 \neq 0$ . Since  $x_0$  is a scalar, from lemma 5.2 the scalar game  $(P^{\mathbb{LQ}})$  has a Pareto solution for every initial state. Let  $x_0 \neq 0$  be fixed. Let  $\alpha$ ,  $z^*(t)$  and  $p(t)$ , with a corresponding solution  $v^*(x_0)$ , solve (30). Due to linearity we see that for the same choice of  $\alpha$ ,  $\mu x_0$ ,  $\mu z^*(t)$  and  $\mu p(t)$  also solves (30). This means (30) has a solution for all  $x_0$ . Since,  $v \in L_{2,s}^+(x_0, z, \tilde{A})$  we have  $\lim_{t \rightarrow \infty} z(t) = 0$  for all  $x_0$ . Following the discussion above  $G_\alpha$  must have an eigenvalue with negative real part. Which means  $G_\alpha$  has no eigenvalues on the imaginary axis and there exists a stable graph subspace. So, the (ARE) has a real stabilizing solution from proposition 5.1. Again from lemma 5.4 this means that  $u^*$  minimizes the weighted sum objective with  $\alpha$  as the weight vector for all initial states  $x_0$ .  $\square$

**Remark 9.** a) As already noticed in the proof of lemma 5.5 it follows directly from lemma 5.2 that, in case the scalar game has a Pareto solution for some initial state different from zero, the game (27, 28) has a Pareto solution for every initial state.

- b) From the proof of lemma 5.5 we can in fact conclude the following result. If for  $x_0 \neq 0$  there exists a Pareto solution and (30) has a solution with the choice of  $\vec{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_N) \in \mathcal{P}_N$ , then the scalar optimization problem  $\min_{\sum_{i \in \bar{N}} \bar{\alpha}_i J_i}$  subject to (28) has a solution for all initial states  $x_0 \in \mathbb{R}$ . In other words all candidates obtained from theorem 5.1 are indeed Pareto solutions.

**Theorem 5.6.** *Consider the scalar system with  $R_i > 0$ ,  $i \in \bar{N}$  and  $v \in L_{2,s}^+(x_0, z, \tilde{A})$ . Then for some  $x_0 \neq 0$ /every  $x_0$  (27, 28) has a Pareto optimal control  $\tilde{u}^*(x_0)$  if and only if there exists  $\vec{\alpha} \in \mathcal{P}_N$  such that for every  $x_0$   $\min_{\sum_{i \in \bar{N}} \alpha_i J_i}$  subject to (28) has a solution.*

*Proof.*  $\Rightarrow$  In particular, it follows that if (27, 28) has a Pareto optimal solution for some  $x_0 \neq 0$  then lemma 5.5 yields the advertized result.

$\Leftarrow$  Since  $R_\alpha > 0$ , if there exists an  $\vec{\alpha} \in \mathcal{P}_N$  such that for every  $x_0$   $\min_{\sum_{i \in \bar{N}} \alpha_i J_i}$  has a solution then the associated (ARE) has a stabilizing solution. The control strategy thus obtained is unique. Further, this control is indeed Pareto optimal from theorem 5.1.  $\square$

In other words, theorem 5.6 shows that to find all Pareto solutions of the game (27, 28), with arbitrary initial state (or for some initial state different from zero) one has to determine all  $\vec{\alpha} \in \mathcal{P}_N$  for which (ARE) has a stabilizing solution. From remark 9.b and theorem 5.6 we have the following algorithm to find all the Pareto solutions for a scalar game.

**Algorithm 1.** With assumption 4 holding true, consider the scalar system with  $R_i > 0$ ,  $i \in \bar{N}$ . Let us define the index set

$$I(\alpha) := \{ \vec{\alpha} \in \mathcal{P}_N | (\text{ARE}) \text{ has a stabilizing solution } X_\alpha \}.$$

1. To find all Pareto solutions of the game (27, 28) with arbitrary initial state one has to determine the set  $I(\alpha)$ . Then, the set of Pareto optimal controls is given by

$$\tilde{u}^*(t) = -R_\alpha^{-1} (B'X_\alpha + S_\alpha) e^{(\tilde{A} - BR_\alpha^{-1}(B'X_\alpha + S_\alpha))t} x_0 \text{ with } \vec{\alpha} \in I(\alpha), \quad (35)$$

where  $K_\alpha$  solves the corresponding (ARE).

2. To find all Pareto solutions for which the game (27, 28) has a Pareto solution for a fixed initial state  $x_0 \neq 0$  use step (1).
3. If  $x_0 = 0$  and  $I(\alpha)$  is non empty,  $u^* \equiv 0$  is a Pareto optimal control. We know from corollary 5.1 that if we find a Pareto solution different from zero, then all points on the half-line through this solution and zero are Pareto solutions too.

We consider example 6.2 from [7] to illustrate the usage of algorithm 1.

*Example 6.* Consider the situation in which there are two individuals who invest in a public stock of knowledge. Let  $x(t)$  be the stock of knowledge at time  $t$  and  $u_i(t)$  the investment of player  $i$  in public knowledge at time  $t$ . Assume that the stock of knowledge evolves according to the accumulation equation

$$\dot{x}(t) = -\beta x(t) + u_1(t) + u_2(t), \quad x(0) = x_0,$$

where  $\beta$  is the depreciation rate. Assume that each player derives quadratic utility from the consumption of the stock of knowledge and that the cost of investment increases quadratically with the investment effort. That is, the cost function of both players is given by

$$J_i = \int_0^\infty e^{-\theta t} \{ -q_i x^2(t) + r_i u_i^2(t) \} dt.$$

Since the investment efforts are bounded and the system is controllable all Pareto solutions can be obtained using the weighting method. It can be easily verified that the associated (ARE) has a stabilizing solution if

$$\left( \beta + \frac{\theta}{2} \right)^2 - (\alpha q_1 + (1 - \alpha) q_2) \left( \frac{1}{\alpha r_1} + \frac{1}{(1 - \alpha) r_2} \right) > 0.$$

For the choice of model parameters  $\beta = 2$ ,  $\theta = 0.05$ ,  $r_i = q_i = 1$ ,  $i = 1, 2$  the above condition is written as  $\alpha(1 - \alpha) > 0.2380$ ,  $0 < \alpha < 1$ . Applying the algorithm (1) we find  $I(\alpha) = (0.3902, 0.6098)$  and all Pareto efficient controls are given by (35).

Next, we consider an example to illustrate that in general the set of Pareto optimal control actions is not convex.

*Example 7.* Consider the cooperative game with

$$J_1 = \int_0^\infty e^{-\rho t} \left( x^2(t) + \frac{9}{10} u_1^2(t) + \frac{1}{10} u_2^2(t) \right) dt \quad \text{and} \quad J_2 = \int_0^\infty e^{-\rho t} \left( x^2(t) + \frac{1}{10} u_1^2(t) + \frac{9}{10} u_2^2(t) \right) dt$$

$$\text{subject to the system } \dot{x} = \frac{\rho}{2} x + \frac{4}{10} (u_1 + u_2), \quad x(0) = x_0 \neq 0, \quad \rho > 0.$$

Here player  $i$  controls  $u_i$ . Using the algorithm 1 the game has a Pareto optimal solution for those  $\alpha \in [0, 1]$  for which the (ARE)  $-sx_\alpha^2 + 1 = 0$ , where  $s = \frac{16}{(9\alpha - 8)(8\alpha + 1)}$ . Taking  $x_\alpha = \sqrt{s}$  here, we see that for all  $\alpha \in [0, 1]$  there is a Pareto solution. The set of all Pareto optimal controls is given by

$$u_\alpha^*(t) = e^{\rho t/2} v_\alpha^*(t), \quad v_\alpha^*(t) = -\frac{4x_\alpha}{10} \left[ \frac{10}{1+8\alpha} \right] z^*(t), \quad \dot{z}^*(t) = \frac{4}{10} \begin{bmatrix} 1 & 1 \end{bmatrix} v_\alpha^*(t), \quad z^*(0) = x_0 \neq 0$$

For  $\alpha = \frac{1}{4}$  and  $\alpha = \frac{3}{4}$  the Pareto controls are given by  $v_{\frac{1}{4}}^*(t) = \begin{bmatrix} -1.5275 \\ -0.6547 \end{bmatrix} z^*(t)$  and  $v_{\frac{3}{4}}^*(t) = \begin{bmatrix} -0.6547 \\ -1.5275 \end{bmatrix} z^*(t)$  respectively. Now consider the convex combination  $\tilde{u}(t) := \frac{1}{2} u_{\frac{1}{4}}^*(t) + \frac{1}{2} u_{\frac{3}{4}}^*(t)$ . With straight forward calculations it can be verified that this choice of control yields the same cost,  $\tilde{J} = 1.2548$ ,  $i = 1, 2$ , for both players. On the other hand choosing  $\alpha = \frac{1}{2}$  we observe that the Pareto control  $u_{\frac{1}{2}}^*$  yields a lower cost, same for both the players,  $J_{\frac{1}{2}}^* = 1.25 < 1.2548 = \tilde{J}$ . So,  $\tilde{u}$  is not Pareto optimal, and this example demonstrates that in general the set of Pareto optimal controls is not convex.

## 6 Conclusions

In this paper we derived necessary conditions for the existence of Pareto solutions in an infinite horizon cooperative differential game with open loop information structure. We considered non-autonomous and discounted autonomous systems for the analysis. These conditions are in the spirit of the maximum principle. For autonomous systems we derived some necessary conditions for optimality by exploiting the special constraint structure (due to reformulation of Pareto optimality). We gave some weak conditions, related to the extension of finite horizon transversality conditions, under which the necessary conditions for Pareto optimality are same as those of a weighted sum optimal control problem. Furthermore, we derived conditions under which the necessary conditions are also sufficient.

Later, the obtained results are used to analyze the regular indefinite infinite horizon linear quadratic differential game. We showed that if the dynamic system is controllable then all Pareto candidates can be obtained by solving the necessary conditions for optimality of a weighted sum optimal control problem. For the two player case we showed that almost all Pareto solutions can be obtained by using the weighting method even if player's cost functions are not convex. For the  $N$  player scalar case we presented an algorithm to calculate all the Pareto solutions if the initial state differs from zero. This algorithm proceeds by determining the elements in the unit simplex for which the associated weighted algebraic Riccati equation has a solution. We illustrated the subtleties with relevant examples.

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## Appendix

*Proof of Lemma 3.1.* We prove the lemma using a contradiction argument. Suppose  $(x^*(t), t, u^*(t), 1), t \in [0, T]$  is not optimal for the problem  $(P_{it}^p)$  then there exists an admissible pair  $(\hat{Y}(t), \hat{z}(t), \hat{U}(t), \hat{v}(t)), t \in [0, T]$  such that

$$h_i(\hat{z}(T) - T) + \int_0^T \hat{v}(t) e^{-\rho \hat{z}(t)} g_i(\hat{Y}(t), \hat{U}(t)) dt < \int_0^\infty e^{-\rho t} g_i(x^*(t), u^*(t)) dt,$$

$$h_j(\hat{z}(T) - T) + \int_0^T \hat{v}(t) e^{-\rho \hat{z}(t)} g_j(\hat{Y}(t), \hat{U}(t)) dt \leq \int_0^\infty e^{-\rho t} g_j(x^*(t), u^*(t)) dt, \forall j \in \bar{N} \setminus \{i\}.$$

Since  $\hat{v}(t) \in [1/2, \infty)$ ,  $\hat{z}(t)$  is an increasing function defined on  $[0, T]$  so by the inverse function theorem  $\hat{z}(t)$  is invertible on  $[0, \hat{z}(T)]$ . We define  $\hat{x}(s) = \hat{Y}(\hat{z}^{-1}(s))$  and  $\hat{u}(s) = \hat{U}(\hat{z}^{-1}(s))$  for  $s \in [0, \hat{z}(T)]$  and observe that  $\hat{x}(0) = \hat{Y}(\hat{z}^{-1}(0)) = x_0$  and  $\hat{x}(\hat{z}(T)) = \hat{Y}(\hat{z}^{-1}(\hat{z}(T))) = \hat{Y}(T) = x^*(T)$ . Further, we have  $\hat{x}(s)$  defined on  $s \in [0, \hat{z}(T)]$  as:

$$\hat{x}(\hat{z}(t)) = x_0 + \int_0^t \hat{Y}(s) ds = x_0 + \int_0^T \hat{v}(t) f(Y(t), U(t)) dt.$$

Taking  $s = z(t)$  we have

$$\hat{x}(s) = x_0 + \int_0^s f(Y(z^{-1}(s)), U(z^{-1}(s))) ds = x_0 + \int_0^s f(\hat{x}(s), \hat{u}(s)) ds,$$

for  $s \in [0, \hat{z}(T)]$ . Since  $\hat{x}(s)$  satisfies the above integral equation we have  $\dot{\hat{x}}(s) = f(\hat{x}(s), \hat{u}(s))$ ,  $\hat{x}(0) = x_0$ ,  $s \in [0, \hat{z}(T)]$ . Next, for  $s > \hat{z}(T)$ , we define  $\hat{x}(s) = x^*(s - \hat{z}(T) + T)$  and  $\hat{u}(s) = u^*(s - \hat{z}(T) + T)$ . Then we observe that  $\dot{\hat{x}}(s) = f(\hat{x}(s), \hat{u}(s))$  with  $\hat{x}(\hat{z}(T)) = x^*(T)$ . Clearly the pair  $(\hat{x}(s), \hat{u}(s))$ ,  $s \in [0, \infty)$  is admissible for problem  $(P_i^\rho)$  and satisfies the following conditions

$$\begin{aligned} \int_0^\infty e^{-\rho t} g_i(\hat{x}(t), \hat{u}(t)) dt &< \int_0^\infty e^{-\rho t} g_i(x^*(t), u^*(t)) dt, \\ \int_0^\infty e^{-\rho t} g_j(\hat{x}(t), \hat{u}(t)) dt &\leq \int_0^\infty e^{-\rho t} g_j(x^*(t), u^*(t)) dt, \quad \forall j \in \bar{N} \setminus \{i\}, \end{aligned}$$

which clearly violates the optimality of  $(x^*(t), u^*(t))$  for the problem  $(P_i^\rho)$ .  $\square$

*Proof of Theorem 3.2.*  $(P_{i_T}^\rho)$  is a mixed endpoint constrained finite horizon optimal control problem. We first define the Hamiltonian  $H_{i_T}(\vec{\lambda}_{i_T}, z(t), Y(t), v(t), U(t))$  as:

$$H_{i_T}(\cdot) = v(t)e^{-\rho z(t)} \left( \lambda_{i_T}^0 g_i(Y(t), U(t)) + \sum_{j \in \bar{N} \setminus \{i\}} \mu_{j_T}^i(t) g_j(Y(t), U(t)) \right) + v(t) \lambda_i'(t) f(Y(t), U(t)) + v(t) \gamma_i(t). \quad (36)$$

The necessary conditions are: there exist  $\lambda_{i_T}^0 \in \mathbb{R}_+$ ,  $\mu_{j_T}^i(t) \in \mathbb{R}$ ,  $\lambda_{i_T}(t) \in \mathbb{R}^n$ ,  $\gamma_{i_T}(t) \in \mathbb{R}$  such that for almost every  $t \in [0, T]$  (the partial derivatives of the Hamiltonian given below are evaluated at the optimal pair  $((x^*(t), t), (u^*(t), 1))$ ):

$$\begin{aligned} \dot{\lambda}_{i_T}(t) &= -(H_{i_T})_Y \\ &= -f'_x(x^*(t), u^*(t)) \lambda_{i_T}(t) - \lambda_{i_T}^0 e^{-\rho t} g_{i_x}(x^*(t), u^*(t)) - \sum_{j \in \bar{N} \setminus \{i\}} \mu_{j_T}^i(t) e^{-\rho t} g_{j_x}(x^*(t), u^*(t)), \quad \lambda_{i_T}(T) \text{ free} \\ \mu_{j_T}^i(t) &= -(H_{i_T})_{\tilde{y}_j^i}, \quad \mu_{j_T}^i(T) \geq 0, \quad \mu_{j_T}^i(T) (\tilde{x}_j(T) + h_j(0) - \tilde{x}_j^*) = 0, \quad \forall j \in \bar{N} \setminus \{i\} \\ \dot{\gamma}_{i_T}(t) &= -(H_{i_T})_z, \quad \gamma_{i_T}(T) = \lambda_{i_T}^0 h_{i_x}(0) + \mu_{j_T}^i(T) h_{j_x}(0) \\ &\quad (\lambda_{i_T}^0, \{\mu_{j_T}^i(t), j \in \bar{N} \setminus \{i\}\}, \lambda_{i_T}(t), \gamma_{i_T}(t)) \neq (0, \dots, 0). \end{aligned}$$

Since  $(H_{i_T})_{\tilde{x}_j^i} = 0$ , we have  $\mu_{j_T}^i(t) = \mu_{j_T}^i \geq 0$ ,  $\forall j \in \bar{N} \setminus \{i\}$ . Let  $\vec{\lambda}_{i_T} = (\mu_{1_T}^i, \dots, \mu_{i-1_T}^i, \lambda_{i_T}^0, \mu_{i+1_T}^i, \dots, \mu_{N_T}^i)$ . Then  $\vec{\lambda}_{i_T} \in \mathbb{R}_+^N$ . Next we show by contradiction that also from the necessary conditions  $(\vec{\lambda}_{i_T}, \lambda_{i_T}(0)) \neq (0, 0)$ . For, if  $(\vec{\lambda}_{i_T}, \lambda_{i_T}(0)) = (0, 0)$  then the necessary conditions give that

$$\dot{\lambda}_{i_T}(t) = -f'_x(x^*(t), u^*(t)) \lambda_{i_T}(t)$$

which results in  $\lambda_{i_T}(t) = 0$  for  $t \in [0, T]$ . Further,  $\vec{\lambda}_{i_T} = 0$  leads to  $\gamma_{i_T}(t) = 0$  for  $t \in [0, T]$  which violates the necessary condition  $(\vec{\lambda}_{i_T}, \lambda_{i_T}(t), \gamma_{i_T}(t)) \neq (0, 0, 0)$  for all  $0 \leq t \leq T$ . Since  $\lambda_{i_T}(T)$  is free, we choose (without loss of generality)  $\lambda_{i_T}(0)$  such that  $\|(\vec{\lambda}_{i_T}, \lambda_{i_T}(0))\| = 1$ . The adjoint variable  $\lambda_{i_T}(t)$  satisfies:

$$\dot{\lambda}_{i_T}(t) = -f'_x(x^*(t), u^*(t)) \lambda_{i_T}(t) - e^{-\rho t} G_x(\vec{\lambda}_{i_T}, x^*(t), u(t)), \quad \lambda_{i_T}(0) = l_{i_T}^0, \quad (37)$$

whereas  $\gamma_{i_T}(t)$  satisfies

$$\dot{\gamma}_{i_T}(t) = -(H_{i_T})_z = \rho G(\vec{\lambda}_{i_T}, x^*(t), u^*(t)). \quad (38)$$

From the definition of  $h(\cdot)$  we have:

$$\begin{aligned} \gamma_{i_T}(T) &= \rho \lambda_{i_T}^0 \int_T^\infty e^{-\rho t} g_i(x^*(t), u^*(t)) dt + \rho \sum_{j \in \bar{N} \setminus \{i\}} \mu_{j_T}^i \int_T^\infty e^{-\rho t} g_j(x^*(t), u^*(t)) dt \\ &= \rho \int_T^\infty e^{-\rho t} G(\vec{\lambda}_{i_T}, x^*(t), u^*(t)) dt. \end{aligned} \quad (39)$$



The Hamiltonian is linear in  $v(t)$  and the minimum w.r.t  $(U(t), v(t))$  on the set  $\mathcal{U} \times [1/2, \infty)$  is attained at  $(u^*(t), 1)$ . The minimum of the Hamiltonian w.r.t  $v(t)$  for  $U(t) = u^*(t)$  is achieved at an interior point of  $[1/2, \infty)$ , so we have:

$$e^{-\rho t} G(\vec{\lambda}_{i_T}, x^*(t), u^*(t)) + \lambda'_{i_T}(t) f(x^*(t), u^*(t)) + \gamma_{i_T}(t) = 0, \quad \forall t \in [0, T]. \quad (40)$$

The minimum of the Hamiltonian w.r.t  $U(t)$  is independent of  $v(t)$  (positive scaling) and does not depend on the term  $\gamma(t)v(t)$ . So, the minimum of  $H_{i_T}$  w.r.t  $U(t)$  at  $v(t) = 1$  is achieved at

$$\begin{aligned} u^*(t) &= \underset{U(t)}{\operatorname{argmin}} H_{i_T}(\vec{\lambda}_{i_T}, t, x^*(t), z(t) = t, U(t), v(t) = 1, \lambda_{i_T}(t), \gamma_{i_T}(t)) \\ &= \underset{u(t)}{\operatorname{argmin}} \left( e^{-\rho t} G(\vec{\lambda}_{i_T}, x^*(t), u(t)) + \lambda'_{i_T}(t) f(x^*(t), u(t)) \right), \quad \forall t \in [0, T]. \end{aligned} \quad (41)$$

Now, consider an increasing sequence  $\{T_k\}_{k \in \mathbb{N}}$  such that  $\lim_{k \rightarrow \infty} T_k = \infty$ . We can associate an optimal control problem  $(P_{i_{T_k}})$  with each  $T_k$  such that the necessary conditions as discussed above hold true. Then there exists a sequence  $\left\{ \left( \vec{\lambda}_{i_{T_k}}, l_{i_{T_k}}^0 \right) \right\}$  such that  $\vec{\lambda}_{i_{T_k}} \in \mathbb{R}_+^N$  and  $\left\| \left( \vec{\lambda}_{i_{T_k}}, l_{i_{T_k}}^0 \right) \right\| = 1$ . We know from the Bolzano-Weierstrass theorem that every bounded sequence has a convergent subsequence. Using the same indices for such a subsequence we infer that there exists  $\vec{\lambda}_i \in \mathbb{R}_+^N$  and  $l_i^0$  such that

$$\lim_{k \rightarrow \infty} \vec{\lambda}_{i_{T_k}} = \vec{\lambda}_i \in \mathbb{R}_+^N \quad \text{and} \quad \lim_{k \rightarrow \infty} l_{i_{T_k}}^0 = l_i^0 \quad \text{such that} \quad \left\| \left( \vec{\lambda}_i, l_i^0 \right) \right\| = 1. \quad (42)$$

We observe that (37) is a linear ODE. So we can write  $\lambda_{i_{T_k}}(t)$  as:

$$\lambda_{i_{T_k}}(t) = \Phi_{-f'_x}(t, 0) l_{i_{T_k}}^0 - \int_0^t \Phi_{-f'_x}(t, s) e^{-\rho s} G_x(\vec{\lambda}_{i_T}, x^*(s), u^*(s)) ds, \quad (43)$$

where  $\Phi_{-f'_x}(t, s)$  is the state transition matrix associated with  $\dot{z}(t) = -f'_x(x^*(t), u^*(t)) z(t)$ . Since the weights of  $\vec{\lambda}_{i_{T_k}}$  appear linearly in  $G_x(\cdot)$  as  $k \rightarrow \infty$ ,  $\lambda_i(t)$  satisfies the differential equation (21b). A similar argument holds for  $\gamma_{i_{T_k}}(t)$ , condition (40) and (41) resulting in (21c), (21e) and (21d) respectively.  $\square$

Proof of Corollary 3.2. From the necessary conditions (21d) and (21e) of theorem 3.2 we have

$$e^{-\rho t} G(\vec{\lambda}_{i_T}, x^*(t), u(t)) + \lambda'_i(t) f(x^*(t), u(t)) \geq e^{-\rho t} G(\vec{\lambda}_i, x^*(t), u^*(t)) + \lambda'_i(t) f(x^*(t), u^*(t)) = -\gamma_i(t). \quad (44)$$

By assumption 2 we have  $\lambda'_i(t) f(x^*(t), u(t)) \geq -\gamma_i(t)$ . Next define  $q(t)$  as follows:

$$q(t) = \frac{\lambda_i(t)}{\max\{1, \|\lambda_i(t)\|\}}. \quad \text{Since } \|q(t)\| \leq 1 \text{ we have } \limsup_{t \rightarrow \infty} \|q(t)\| = l.$$

If  $l = 0$  there is nothing to prove. So assume  $l > 0$  and consider a sequence  $\{t_n\}$  converging to infinity such that  $\|q(t_n)\| > l/2$ . Since there exists  $u(t) \in \mathcal{U}$  such that  $\mathcal{B}_{\delta > 0}^6 \subset f(x^*(t), u(t))$ , there exists an  $\varepsilon > 0$  such that  $\frac{2\varepsilon}{l} < \delta$ . So, there exists  $u_n(t_n) \in \mathcal{U}$  such that  $f(x^*(t_n), u_n(t_n)) = -(2\varepsilon/l) q(t_n)$ . Since,  $\lim_{t \rightarrow \infty} \gamma_i(t) = 0$  we take the above sequence  $\{t_n\}$  such that  $-l\varepsilon/2 \leq -\gamma(t_n) \leq l\varepsilon/2$ . Collecting all the above we have:

$$\begin{aligned} \lambda'_i(t_n) f(x^*(t_n), u_n(t_n)) &= -\max\{1, \|\lambda_i(t_n)\|\} (2\varepsilon/l) \|q(t_n)\|^2 \geq -\gamma_i(t_n) \\ \gamma_i(t_n) &\geq \max\{1, \|\lambda_i(t_n)\|\} (2\varepsilon/l) \|q(t_n)\|^2 > l\varepsilon/2. \end{aligned}$$

Clearly, this is a contradiction and thus  $\lim_{t \rightarrow \infty} \lambda_i(t) = 0$ .  $\square$

Proof of Corollary 3.3. From the assumption 3 there exist constants  $c_4 \geq 0$ ,  $c_5 \geq 0$ ,  $c_6 \geq 0$  and  $c_7 \geq 0$  such that:

$$e^{-\rho t} |g_i(x(t), u(t))| \leq e^{-\rho t} (c_4 + c_5 \|x(t)\|^{r+1}) \leq c_6 e^{-\rho t} + c_7 e^{-(\rho - (r+1)\lambda)t}.$$

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<sup>6</sup>Unit ball in  $\mathbb{R}^n$  of radius  $\delta > 0$ .

Since  $\rho > (1+r)\lambda$ , the player's costs  $J_i(u)$  converge for every admissible pair  $(x(t), u(t))$ . For the problem  $(P_i)$  we rewrite  $\lambda_i(t)$  (from (21b)) as follows:

$$\begin{aligned}
\lambda_i(t) &= \Phi_{-f'_x}(t, 0) l_i^0 - \int_0^t e^{-\rho s} \Phi_{-f'_x}(t, s) G_x(x^*(s), u^*(s)) ds \\
&= \Phi_{-f'_x}(t, 0) \left( l_i^0 - \int_0^t e^{-\rho s} \Phi_{-f'_x}(0, s) G_x(x^*(s), u^*(s)) ds \right) \\
&\quad \left( \text{we know } \Phi_{-f'_x}(t, s) = \left( \Phi_{f'_x}^{-1}(t, s) \right)' = \Phi'_{f'_x}(s, t) \right) \\
&= \Phi_{-f'_x}(t, 0) \left( l_i^0 - \int_0^t e^{-\rho s} \Phi'_{f'_x}(s, 0) G_x(x^*(s), u^*(s)) ds \right).
\end{aligned}$$

The norm of  $\lambda_i(t)$  is bounded as:

$$\begin{aligned}
\|\lambda_i(t)\| &\leq \|\Phi_{-f'_x}(t, 0)\| \left( \|l_i^0\| + \int_0^t e^{-\rho s} \|\Phi_{f'_x}(s, 0)\| \|G_x(x^*(s), u^*(s))\| ds \right) \\
&\quad (\text{from (3b) there exist a } c_8 \geq 0, \text{ a } c_9 \geq 0 \text{ such that}) \\
&\leq \|\Phi_{-f'_x}(t, 0)\| \left( \|l_i^0\| + \int_0^t e^{-\rho s} (c_8 e^{\lambda s} + c_9 e^{(1+r)\lambda s}) ds \right) \\
&\quad (l_i^0 \text{ is bounded, so there exist a } c_{10} \geq 0, \text{ a } c_{11} \geq 0 \text{ and } c_{12} \geq 0) \text{ such that} \\
&\leq \|\Phi_{-f'_x}(t, 0)\| \left( c_{10} + c_{11} e^{-(\rho-\lambda)t} + c_{12} e^{-(\rho-(1+r)\lambda)t} \right).
\end{aligned}$$

Let  $\phi^0(t)$  and  $\phi_0(t)$  denote the largest and smallest eigenvalues of the Hermitian part<sup>7</sup> of  $-f'_x(x^*(t), u^*(t))$ . By assumption  $-f'_x(x^*(t), u^*(t))$  is bounded and has strictly negative eigenvalues so we have  $-\infty < \phi_0(t) \leq \phi^0(t) < 0$ ,  $\forall t \geq 0$ . From [13, lemma 4.2] we have:

$$\exp \left( \int_0^T \mu_0(s) ds \right) \leq \|\Phi_{-f'_x}(t, 0)\| \leq \exp \left( \int_0^T \mu^0(s) ds \right).$$

Since  $\mu^0(s) < 0$  for all  $s \geq 0$  we have  $\lim_{t \rightarrow \infty} \|\Phi_{-f'_x}(t, 0)\| = 0$ . By assumption  $\rho > (1+r)\lambda$  so  $\lim_{t \rightarrow \infty} \lambda_i(t) = 0$  follows directly.  $\square$

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<sup>7</sup>The Hermitian part of matrix  $A$  is defined here as  $A^H = \frac{1}{2}(A + A')$ .